A GENERALIZATION OF THE $Z^*$-THEOREM

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Abstract. Glaubermann’s $Z^*$-theorem and analogous statements for odd primes show that, for any prime $p$ and any finite group $G$ with Sylow $p$-subgroup $S$, the centre of $G$ is determined by the fusion system $\mathcal{F}_S(G)$. Building on these results we show a statement that can be considered as a generalization: For any normal subgroup $N$ of $G$, the centralizer $C_S(N)$ is expressed in terms of the fusion system $\mathcal{F}_S(G)$ and its normal subsystem induced by $N$.

Throughout $p$ is a prime. Glaubermann’s $Z^*$-theorem \cite{Glaubermann} and its generalization to odd primes, which is shown using the classification of finite simple groups (see \cite{Aschbacher} and \cite{Glaubermann}), can be reformulated as follows:

**Theorem A.** Let $G$ be a finite group with $O_{p'}(G) = 1$, and $S \in \text{Syl}_p(G)$. Then $Z(G) = Z(\mathcal{F}_S(G))$.

We refer the reader here to \cite{Aschbacher} for basic definitions and results regarding fusion systems; see in particular Definitions I.4.1 and I.4.3 for the definition of central subsystems and the centre $Z(\mathcal{F})$.

Given a saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$ and a normal subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leq S$, Aschbacher \cite{Aschbacher} (6.7)(1)] showed that the set of subgroups $X$ of $C_S(T)$ with $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ has a largest member $C_S(\mathcal{E})$. He furthermore constructed a normal subsystem $C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$, the centralizer of $\mathcal{E}$ in $\mathcal{F}$; see [1 Chapter 6].

Note that $C_S(\mathcal{E})$ depends not only on $S$ and $\mathcal{E}$ but also on the fusion system $\mathcal{F}$ in which both $S$ and $\mathcal{E}$ are contained.

The definition of $C_S(\mathcal{E})$ generalizes the definition of $Z(\mathcal{F})$ since $C_S(\mathcal{F}) = Z(\mathcal{F})$. Moreover, for every normal subgroup $H$ of a finite group $G$ with Sylow $p$-subgroup $S$, $\mathcal{F}_{S \cap H}(H)$ is a normal subsystem of $\mathcal{F}_S(G)$ by [2 I.6.2]. Thus, the following theorem, which we prove in this short note, can be seen as a generalization of Theorem A.

**Theorem B.** Let $G$ be a finite group and $S$ be a Sylow $p$-subgroup of $G$. Let $H \leq G$ with $O_{p'}(H) = 1$. Then $C_S(\mathcal{F}_{S \cap H}(H)) = C_S(H)$.

In the statement of Theorem B it is understood that $C_S(\mathcal{F}_{S \cap H}(H))$ is formed inside of $\mathcal{F}_S(G)$. The result says in other words that, under the hypothesis of Theorem B, for any $X \leq S$ with $\mathcal{F}_{S \cap H}(H) \subseteq C_{\mathcal{F}_S(G)}(X)$, we have $X \leq C_S(H)$. This is not true if one drops the assumption that $H$ is normal in $G$ as the following example shows: Let $G := G_1 \times G_2$ with $G_1 \cong G_2 \cong S_3$. Set $p = 3$, $S = O_3(G)$, $S_1 := O_3(G_1)$ and let $R$ be a subgroup of $G$ of order 2 which acts fixed point freely on $S$. Set $H := S_1 \rtimes R$. Then $S_1 = S \cap H \in \text{Syl}_3(H)$ and $\mathcal{F}_{S_1}(H) = \mathcal{F}_{S_1}(G_1) \subseteq C_{\mathcal{F}_S(G)}(S_2)$ as $S_2 = C_S(G_1)$. However, $S_2 \not\subseteq C_S(H)$ by the choice of $R$.

Theorem B was conjectured by the second author of this paper in \cite{Henke}. Our proof of Theorem B builds on Theorem A and the reduction uses only elementary group
theoretical results. Essential is the following lemma, whose proof is self-contained apart from using the conjugacy of Hall-subgroups in solvable groups.

**Lemma 1.** Let $G$ be a finite group with Sylow $p$-subgroup $S$ and a normal subgroup $H$. Let $P \leq S$ such that $P \cap H$ is centric in $\mathcal{F}_{S \cap H}(H)$. Then for every $p'$-element $\varphi \in \text{Aut}_G(P)$ with $[P, \varphi] \leq P \cap H$ and $\varphi|_{P \cap H} \in \text{Aut}_H(P)$, we have $\varphi \in \text{Aut}_H(P)$.

**Proof.** This is [1, Proposition 3.1].

**Proof of Theorem 2.** We assume the hypothesis of Theorem 2. Furthermore, we set $\mathcal{F} := \mathcal{F}_S(G)$, $T := S \cap H$ and $\mathcal{E} := \mathcal{F}_T(H)$. If a homomorphism $\varphi$ between subgroups $A$ and $B$ of $T$ is induced by conjugation with an element $h \in H$, then $\varphi$ extends to $c_h : AC_S(H) \to BC_S(H)$ and $c_h$ restricts to the identity on $C_S(H)$. Thus $\mathcal{E} \subseteq C_T(C_S(H))$, so by the definition of $C_S(\mathcal{E})$, we have $C_S(H) \subseteq C_S(\mathcal{E})$. To prove the converse inclusion, choose $t \in C_S(\mathcal{E})$. Define:

$$G_0 := H(t) \quad \text{and} \quad S_0 := T(t),$$

so that plainly $S_0$ is a Sylow $p$-subgroup of $G_0$ and $\mathcal{F}_0 := \mathcal{F}_{S_0}(G_0)$ is a saturated fusion system on $S_0$. Note also that $O_{p'}(G_0) = 1$ as $O^p(G_0) = O^p(H)$ and $O_{p'}(H) = 1$ by assumption.

By Theorem 1, $Z(\mathcal{F}_0) = Z(G_0) \leq C_S(H)$. It thus suffices to prove $t \in Z(\mathcal{F}_0)$. As $t \in C_S(\mathcal{E}) \leq C_S(T)$, $t \in Z(S_0)$. Let $P$ be a subgroup of $S_0$ which is centric radical and fully normalized in $\mathcal{F}_0$. Then $t \in Z(S_0) \leq C_{S_0}(P) \leq P$. It is sufficient to prove $[t, \text{Aut}_{\mathcal{F}_0}(P)] = 1$. For as $P$ is arbitrary, Alperin's fusion theorem [1, Theorem 3.6] implies then $t \in Z(\mathcal{F}_0)$. As $P$ is fully $\mathcal{F}_0$-normalized, $\text{Aut}_{S_0}(P) \leq \text{Syl}_p(\text{Aut}_{\mathcal{F}_0}(P))$ and thus $\text{Aut}_{\mathcal{F}_0}(P) = \text{Aut}_{S_0}(P)O^p(\text{Aut}_{\mathcal{F}_0}(P))$. Note that $[t, \text{Aut}_{S_0}(P)] = 1$ as $t \in Z(S_0)$. Hence, it is enough to prove

$$[t, O^p(\text{Aut}_{\mathcal{F}_0}(P))] = 1.$$

Let $\varphi \in \text{Aut}_{\mathcal{F}_0}(P)$ be a $p'$-element. Since $O^p(H) = O^p(G_0)$, we have $O^p(\text{Aut}_{\mathcal{F}_0}(P)) = O^p(\text{Aut}_H(P))$. In particular, $\varphi \in \text{Aut}_H(P)$ and thus $\varphi|_{P \cap T} \in \text{Aut}_H(P \cap T) = \text{Aut}_T(P \cap T)$. As $t \in P \leq S_0 = T(t)$, we have $P = (P \cap T)\langle t \rangle$. Moreover, $t \in C_S(\mathcal{E})$ implies that $\mathcal{E} \subseteq C_T(t)$). Hence, $\varphi|_{P \cap T}$ extends to $\psi \in \text{Aut}_T(P)$ with the property that $t \psi = t$. Note that $o(\psi) = o(\varphi|_{P \cap T})$ and thus $\psi$ is a $p'$-element as $\varphi$ has order prime to $p$. Moreover, plainly $[P, \psi] \leq P \cap T$ and $\psi|_{P \cap T} = \varphi|_{P \cap T} \in \text{Aut}_H(P)$. Since $\mathcal{E} \subseteq \mathcal{F}_0$, $P \cap T$ is $\mathcal{E}$-centric by [1, 7.18]. Now it follows from Lemma 1 that $\psi \in \text{Aut}_H(P)$. Thus, $\chi := \varphi \circ \psi^{-1} \in \text{Aut}_H(P) \leq \text{Aut}_{\mathcal{F}_0}(P)$. Clearly $\chi|_{P \cap T} = \text{Id}$ as $\psi$ extends $\varphi|_{P \cap T}$. Moreover, using that $H$ is normal in $G$, we obtain $[P, \chi] \leq [P, \text{Aut}_H(P)] = [P, N_H(P)] \leq P \cap H = P \cap T$. Hence, by [1, Lemma A.2], $\chi \in C_{\text{Aut}_{\mathcal{F}_0}(P)}(P) \cap C_{\text{Aut}_{\mathcal{F}_0}(P)}(P \cap T) = O_p(\text{Aut}_{\mathcal{F}_0}(P)) = \text{Inn}(P)$ as $P$ is radical in $\mathcal{F}_0$. As $\text{Inn}(P) \leq \text{Aut}_{S_0}(P)$ and $[t, \text{Aut}_{S_0}(P)] = 1$, it follows $t \chi = t$ by the choice of $\psi$, also $t \psi = t$ and consequently $t \varphi = t$. Since $\varphi$ was chosen to be an arbitrary $p'$-element in $\text{Aut}_{\mathcal{F}_0}(P)$ and $O^p(\text{Aut}_{\mathcal{F}_0}(P))$ is the subgroup generated by these elements, it follows that $[t, O^p(\text{Aut}_{\mathcal{F}_0}(P))] = 1$. As argued above, this yields the assertion.

**References**


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