The limits of large cardinal compatibility for □

Andrew Brooke-Taylor

University of Bristol

Joint work with Sy-David Friedman

Kurt Gödel Research Center for Mathematical Logic
Universität Wien

12 July 2011
Recall Jensen’s □ principle:

**Definition**

*For any cardinal* \( \alpha \), a □\( \alpha \)-sequence is a sequence \( \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \) such that for every \( \beta \in \alpha^+ \cap \text{Lim} \),

- \( C_\beta \) is a closed unbounded subset of \( \beta \),
- \( \text{ot}(C_\beta) \leq \alpha \),
- for any \( \gamma \in \text{lim}(C_\beta) \), \( C_\gamma = C_\beta \cap \gamma \).

*We say □\( \alpha \) holds if there exists a □\( \alpha \)-sequence.*
Recall Jensen’s □ principle:

**Definition**

For any cardinal \( \alpha \), a □\( \alpha \)-sequence is a sequence \( \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \) such that for every \( \beta \in \alpha^+ \cap \text{Lim} \),

- \( C_\beta \) is a closed unbounded subset of \( \beta \),
- \( \text{ot}(C_\beta) \leq \alpha \),
- for any \( \gamma \in \text{lim}(C_\beta) \), \( C_\gamma = C_\beta \cap \gamma \).

We say □\( \alpha \) holds if there exists a □\( \alpha \)-sequence.

□\( \alpha \) is really more a property of \( \alpha^+ \) than of \( \alpha \).
Recall Jensen’s □ principle:

**Definition**

For any cardinal \( \alpha \), a □\( \alpha \)-sequence is a sequence \( \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \) such that for every \( \beta \in \alpha^+ \cap \text{Lim} \),

- \( C_\beta \) is a closed unbounded subset of \( \beta \),
- \( \text{ot}(C_\beta) \leq \alpha \),
- for any \( \gamma \in \text{lim}(C_\beta) \), \( C_\gamma = C_\beta \cap \gamma \).

We say □\( \alpha \) holds if there exists a □\( \alpha \)-sequence.

□\( \alpha \) is really more a property of \( \alpha^+ \) than of \( \alpha \).

In particular, we can (and will) force □\( \alpha \) to hold without adding any new subsets to \( \alpha \).
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:

- Solovay showed that □$_\alpha$ fails for all $\alpha$ greater than or equal to a supercompact cardinal.
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:

- Solovay showed that □$\alpha$ fails for all $\alpha$ greater than or equal to a supercompact cardinal.
- Jensen (? Burke?) showed that subcompactness of a cardinal $\kappa$ is sufficient to make □$\kappa$ fail.
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:

- Solovay showed that □$_\alpha$ fails for all $\alpha$ greater than or equal to a supercompact cardinal.
- Jensen (?) Burke?) showed that subcompactness of a cardinal $\kappa$ is sufficient to make □$_\kappa$ fail.
- On the other hand, Cummings and Schimmerling have shown that □$_\kappa$ can hold at a cardinal $\kappa$ which is 1-extendible, a notion just short of subcompactness.
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:

- Solovay showed that $□_\alpha$ fails for all $\alpha$ greater than or equal to a supercompact cardinal.
- Jensen (? Burke?) showed that subcompactness of a cardinal $\kappa$ is sufficient to make $□_\kappa$ fail.
- On the other hand, Cummings and Schimmerling have shown that $□_\kappa$ can hold at a cardinal $\kappa$ which is 1-extendible, a notion just short of subcompactness.
- Moreover, Schimmerling and Zeman have shown that in Jensen extender models, $□_\kappa$ fails if and only if $\kappa$ is subcompact.
□ and large cardinals: what was known

Like the axiom $V = L$, but unlike many other properties of $L$ such as GCH and the existence of morasses, □ is inconsistent with sufficiently strong large cardinal axioms:

- Solovay showed that $\square_\alpha$ fails for all $\alpha$ greater than or equal to a supercompact cardinal.
- Jensen (? Burke?) showed that subcompactness of a cardinal $\kappa$ is sufficient to make $\square_\kappa$ fail.
- On the other hand, Cummings and Schimmerling have shown that $\square_\kappa$ can hold at a cardinal $\kappa$ which is 1-extendible, a notion just short of subcompactness.
- Moreover, Schimmerling and Zeman have shown that in Jensen extender models, $\square_\kappa$ fails if and only if $\kappa$ is subcompact.
  - but Jensen extender models can’t tolerate large cardinals much bigger than that anyway.
Generalising Jensen’s subcompactness

Recall that for any cardinal $\alpha$, we denote by $H_\alpha$ the set of all sets whose transitive closure has cardinality strictly less than $\alpha$.

**Definition**
For any cardinal $\alpha$, we say that a cardinal $\kappa < \alpha$ is $\alpha$-**subcompact** if for every $A \subseteq H_\alpha$, there exist $\kappa < \bar{\alpha} < \kappa$ and $\bar{A} \subseteq H_{\bar{\alpha}}$ such that there is an elementary embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \rightarrow (H_\alpha, \in, A)$$

with critical point $\bar{\kappa}$ satisfying $\pi(\bar{\kappa}) = \kappa$. 

In this terminology, Jensen’s original notion of subcompactness is $\kappa^+ -$subcompact. Also note that if $\kappa < \beta < \alpha$ and $\kappa$ is $\alpha -$subcompact, then $\kappa$ is $\beta -$subcompact.
Generalising Jensen’s subcompactness

Recall that for any cardinal $\alpha$, we denote by $H_\alpha$ the set of all sets whose transitive closure has cardinality strictly less than $\alpha$.

**Definition**
For any cardinal $\alpha$, we say that a cardinal $\kappa < \alpha$ is $\alpha$-subcompact if for every $A \subseteq H_\alpha$, there exist $\bar{\kappa} < \bar{\alpha} < \kappa$ and $\bar{A} \subseteq H_{\bar{\alpha}}$ such that there is an elementary embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \rightarrow (H_\alpha, \in, A)$$

with critical point $\bar{\kappa}$ satisfying $\pi(\bar{\kappa}) = \kappa$.

In this terminology, Jensen’s original notion of subcompactness is $\kappa^+$-subcompactness.
Generalising Jensen’s subcompactness

Recall that for any cardinal $\alpha$, we denote by $H_\alpha$ the set of all sets whose transitive closure has cardinality strictly less than $\alpha$.

**Definition**

For any cardinal $\alpha$, we say that a cardinal $\kappa < \alpha$ is $\alpha$-*subcompact* if for every $A \subseteq H_\alpha$, there exist $\bar{\kappa} < \bar{\alpha} < \kappa$ and $\bar{A} \subseteq H_{\bar{\alpha}}$ such that there is an elementary embedding

$$
\pi : (H_{\bar{\alpha}}, \in, \bar{A}) \to (H_\alpha, \in, A)
$$

with critical point $\bar{\kappa}$ satisfying $\pi(\bar{\kappa}) = \kappa$.

In this terminology, Jensen’s original notion of subcompactness is $\kappa^+$-subcompactness. Also note that if $\kappa < \beta < \alpha$ and $\kappa$ is $\alpha$-subcompact, then $\kappa$ is $\beta$-subcompact.
How strong is subcompactness?

Following an old argument of Magidor, we get:

**Proposition**

1. If $\kappa$ is $2^{<\alpha}$-supercompact, then $\kappa$ is $\alpha$-subcompact.
2. If $\kappa$ is $(2^{(\lambda^{<\kappa})^+})$-subcompact, then $\kappa$ is $\lambda$-supercompact.

In particular, $\kappa$ is supercompact if and only if $\kappa$ is $\alpha$-subcompact for every $\alpha > \kappa$. 
Theorem

Suppose $\kappa$ is $\alpha^+$-subcompact for some $\alpha \geq \kappa$. Then $\Box_\alpha$ fails.
Theorem

Suppose $\kappa$ is $\alpha^+\text{-subcompact}$ for some $\alpha \geq \kappa$. Then $\square_\alpha$ fails.

Proof (essentially the same as Jensen’s)

Suppose for contradiction that $\kappa$ is $\alpha^+\text{-subcompact}$ but there is a $\square_\alpha$-sequence $C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle$. 
Theorem

Suppose \( \kappa \) is \( \alpha^+ \)-subcompact for some \( \alpha \geq \kappa \). Then \( \square_\alpha \) fails.

Proof (essentially the same as Jensen’s)

Suppose for contradiction that \( \kappa \) is \( \alpha^+ \)-subcompact but there is a \( \square_\alpha \)-sequence \( C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \).

Let \( \pi : (H_{\tilde{\alpha}^+}, \in, \tilde{C}) \to (H_{\alpha^+}, \in, C) \) be an \( \alpha^+ \)-subcompactness embedding with critical point \( \tilde{\kappa} \), \( \pi(\tilde{\kappa}) = \kappa \). Note by elementarity that \( \pi(\tilde{\alpha}) = \alpha \).
Theorem

Suppose \( \kappa \) is \( \alpha^+ \)-subcompact for some \( \alpha \geq \kappa \). Then \( \square_\alpha \) fails.

Proof (essentially the same as Jensen’s)

Suppose for contradiction that \( \kappa \) is \( \alpha^+ \)-subcompact but there is a \( \square_\alpha \)-sequence \( C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \).

Let \( \pi : (H_{\bar{\alpha}^+}, \in, \bar{C}) \to (H_{\alpha^+}, \in, C) \) be an \( \alpha^+ \)-subcompactness embedding with critical point \( \bar{\kappa} \), \( \pi(\bar{\kappa}) = \kappa \). Note by elementarity that \( \pi(\bar{\alpha}) = \alpha \).

Let \( \lambda = \text{sup}(\pi^{\bar{\alpha}^+}) \) and consider the set

\[
D = \text{lim}(C_\lambda) \cap \pi^{\bar{\alpha}^+}.
\]
Theorem

Suppose $\kappa$ is $\alpha^+$-subcompact for some $\alpha \geq \kappa$. Then $\square_\alpha$ fails.

Proof (essentially the same as Jensen’s)

Suppose for contradiction that $\kappa$ is $\alpha^+$-subcompact but there is a $\square_\alpha$-sequence $C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle$.

Let $\pi : (H_{\bar{\alpha}^+}, \in, \bar{C}) \rightarrow (H_{\alpha^+}, \in, C)$ be an $\alpha^+$-subcompactness embedding with critical point $\bar{\kappa}$, $\pi(\bar{\kappa}) = \kappa$. Note by elementarity that $\pi(\bar{\alpha}) = \alpha$.

Let $\lambda = \sup(\pi "\bar{\alpha}^+ ")$ and consider the set

$$D = \text{lim}(C_\lambda) \cap \pi "\bar{\alpha}^+ ".$$

Since $\pi "\bar{\alpha}^+ "$ is $\bar{\kappa}$-closed and unbounded in $\lambda$, $D$ is also unbounded in $\lambda$; in particular, $|D| \geq \bar{\alpha}^+$. 
Theorem

Suppose $\kappa$ is $\alpha^+-$subcompact for some $\alpha \geq \kappa$. Then $\Box_\alpha$ fails.

Proof (essentially the same as Jensen’s)

Suppose for contradiction that $\kappa$ is $\alpha^+$-subcompact but there is a $\Box_\alpha$-sequence $C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle$.

Let $\pi : (H_{\bar{\alpha}^+}, \in, \bar{C}) \rightarrow (H_{\alpha^+}, \in, C)$ be an $\alpha^+$-subcompactness embedding with critical point $\bar{\kappa}$, $\pi(\bar{\kappa}) = \kappa$. Note by elementarity that $\pi(\bar{\alpha}) = \alpha$.

Let $\lambda = \sup(\pi(\bar{\alpha}^+))$ and consider the set

$$D = \lim(C_\lambda) \cap \pi(\bar{\alpha}^+).$$

Since $\pi(\bar{\alpha}^+)$ is $\bar{\kappa}$-closed and unbounded in $\lambda$, $D$ is also unbounded in $\lambda$; in particular, $|D| \geq \bar{\alpha}^+$.

For each $\delta \in D$, $C_\delta$ is an initial segment of $C_\lambda$, which itself has order type at most $\alpha$ (by the definition of square).
Theorem
Suppose \( \kappa \) is \( \alpha^+ \)-subcompact for some \( \alpha \geq \kappa \). Then \( \square_\alpha \) fails.

Proof (essentially the same as Jensen’s)
Suppose for contradiction that \( \kappa \) is \( \alpha^+ \)-subcompact but there is a \( \square_\alpha \)-sequence \( C = \langle C_\beta \mid \beta \in \alpha^+ \cap \text{Lim} \rangle \).
Let \( \pi : (H_{\tilde{\alpha}^+}, \in, \tilde{C}) \rightarrow (H_{\alpha^+}, \in, C) \) be an \( \alpha^+ \)-subcompactness embedding with critical point \( \tilde{\kappa} \), \( \pi(\tilde{\kappa}) = \kappa \). Note by elementarity that \( \pi(\tilde{\alpha}) = \alpha \).
Let \( \lambda = \sup(\pi\text{“}\tilde{\alpha}^+) \) and consider the set
\[
D = \lim(C_\lambda) \cap \pi\text{“}\tilde{\alpha}^+.
\]
Since \( \pi\text{“}\tilde{\alpha}^+ \) is \( \tilde{\kappa} \)-closed and unbounded in \( \lambda \), \( D \) is also unbounded in \( \lambda \); in particular, \( |D| \geq \tilde{\alpha}^+ \).
For each \( \delta \in D \), \( C_\delta \) is an initial segment of \( C_\lambda \), which itself has order type at most \( \alpha \) (by the definition of square).
Thus, \( \{\text{ot}(C_\delta) \mid \delta \in D\} \) is a set of \( \tilde{\alpha}^+ \)-many distinct ordinals less than \( \alpha = \pi(\tilde{\alpha}) \) in the image of \( \pi \).
\( \square \)
Assuming GCH, the previous result is in some sense optimal:

**Theorem (under GCH)**

Let

\[ I = \{ \alpha \mid \exists \kappa \leq \alpha (\kappa \text{ is } \alpha^+-\text{subcompact}) \}. \]

Then there is a cofinality-preserving class forcing \( \mathbb{P} \) such that for any \( \mathbb{P} \)-generic \( G \) the following hold.

1. If \( \kappa < \alpha \) are such that \( V \models \kappa \text{ is } \alpha\text{-subcompact} \), then

\[ V[G] \models \kappa \text{ is } \alpha\text{-subcompact}. \]

In particular, \( I^V[G] = I \).

2. \( \square_\alpha \) holds in \( V[G] \) for all \( \alpha \notin I \).
The proof is to do the natural thing:
The proof is to do the natural thing: a reverse Easton forcing iteration,
The proof is to do the natural thing: a reverse Easton forcing iteration, which

- at stage $\alpha$ for $\alpha \notin I$, forces $\square_\alpha$, 

...
The proof is to do the natural thing: a reverse Easton forcing iteration, which

- at stage $\alpha$ for $\alpha \notin I$, forces $\square_\alpha$,
- at other stages, does nothing (is the trivial forcing).
The proof is to do the natural thing: a reverse Easton forcing iteration, which

- at stage $\alpha$ for $\alpha \notin I$, forces $\square_\alpha$,
- at other stages, does nothing (is the trivial forcing).

We claim that any embedding

$$\pi : (H_{\bar{\alpha}}, \in, \bar{\sigma}) \to (H_\alpha, \in, \sigma)$$

witnessing $\alpha$-subcompactness of some $\kappa$ for $\sigma$ a $\mathbb{P}_\kappa$-name for a subset of $H_{\alpha}[G]$ (and itself an element of $H_\alpha^V$) lifts to

$$\pi' : (H_{\bar{\alpha}}^V[G], \in, \bar{\sigma}_G) \to (H_\alpha^V[G], \in, \sigma_G)$$

$$: \tau_G \mapsto (\pi(\tau))_G.$$
Why just a $\mathbb{P}_\kappa$-name?

Because from stage $\alpha$ of the iteration onward, $\mathbb{P}$ adds no new subsets of $H_\alpha$, and by definition, the forcing iterands are trivial on $[\kappa, \alpha)$. Thus, every new subset of $H_\alpha$ in $V[G]$ is given by a $\mathbb{P}_\kappa$-name.
Why just a $\mathbb{P}_\kappa$-name?

Because from stage $\alpha$ of the iteration onward, $\mathbb{P}$ adds no new subsets of $H_\alpha$, and by definition, the forcing iterands are trivial on $[\kappa, \alpha)$. Thus, every new subset of $H_\alpha$ in $V[G]$ is given by a $\mathbb{P}_\kappa$-name.

Note that by elementarity, $\bar{\sigma}$ is a $\mathbb{P}_{\bar{\kappa}}$-name, and $\mathbb{P}$ is trivial on $[\bar{\kappa}, \bar{\alpha})$. 
Why just a $\mathbb{P}_\kappa$-name?

Because from stage $\alpha$ of the iteration onward, $\mathbb{P}$ adds no new subsets of $H_\alpha$, and by definition, the forcing iterands are trivial on $[\kappa, \alpha)$. Thus, every new subset of $H_\alpha$ in $V[G]$ is given by a $\mathbb{P}_\kappa$-name.

Note that by elementarity, $\bar{\sigma}$ is a $\mathbb{P}_{\bar{\kappa}}$-name, and $\mathbb{P}$ is trivial on $[\bar{\kappa}, \bar{\alpha})$.

Why is $\tau_G \mapsto (\pi(\tau))_G$ well-defined and elementary?

Because

$$p \models \varphi(\tau) \quad \text{iff} \quad \pi(p) \models \varphi(\pi(\tau)),$$

but $\pi$ is the identity below $\kappa$, so $p = \pi(p)$, and in particular $p \in G$ if and only if $\pi(p) \in G$. 

Why just a $\mathbb{P}_\kappa$-name?

Because from stage $\alpha$ of the iteration onward, $\mathbb{P}$ adds no new subsets of $H_\alpha$, and by definition, the forcing iterands are trivial on $[\kappa, \alpha)$. Thus, every new subset of $H_\alpha$ in $V[G]$ is given by a $\mathbb{P}_\kappa$-name.

Note that by elementarity, $\bar{\sigma}$ is a $\mathbb{P}_{\bar{\kappa}}$-name, and $\mathbb{P}$ is trivial on $[\bar{\kappa}, \bar{\alpha})$.

Why is $\tau_G \mapsto (\pi(\tau))_G$ well-defined and elementary?

Because

$$p \models \varphi(\tau) \quad \text{iff} \quad \pi(p) \models \varphi(\pi(\tau)),$$

but $\pi$ is the identity below $\kappa$, so $p = \pi(p)$, and in particular $p \in G$ if and only if $\pi(p) \in G$.

This completes the proof of the Theorem. \hfill \Box
What about other large cardinals?

Maybe other large cardinals also have some impact that we’ve overlooked. If this forcing destroyed other large cardinals, then perhaps that would indicate that our earlier results aren’t so optimal after all.
What about other large cardinals?

Maybe other large cardinals also have some impact that we’ve overlooked. If this forcing destroyed other large cardinals, then perhaps that would indicate that our earlier results aren’t so optimal after all.

It seems that this scenario doesn’t occur (at least for a very large test case):

**Definition**

A cardinal $\kappa$ is $\omega$-superstrong (I2 in the notation of Kanamori) if and only if there is an elementary embedding $j : V \to M$ with critical point $\kappa$ such that, if we let $\lambda = \sup_{n \in \omega} (j^n(\kappa))$, $V_\lambda \subset M$.

**Proposition**

The forcing iteration $P$ of the theorem above preserves all $\omega$-superstrong cardinals.
What about other large cardinals?

Maybe other large cardinals also have some impact that we’ve overlooked. If this forcing destroyed other large cardinals, then perhaps that would indicate that our earlier results aren’t so optimal after all.

It seems that this scenario doesn’t occur (at least for a very large test case):

**Definition**

A cardinal $\kappa$ is $\omega$-superstrong (I2 in the notation of Kanamori) if and only if there is an elementary embedding $j : V \rightarrow M$ with critical point $\kappa$ such that, if we let $\lambda = \sup_{n \in \omega} (j^n(\kappa))$, $V_\lambda \subset M$.

**Proposition**

The forcing iteration $\mathbb{P}$ of the theorem above preserves all $\omega$-superstrong cardinals.

Again, the large cardinal is preserved because the forcing is trivial everywhere that counts.
Stationary reflection

Definition
For regular $\kappa > \lambda$, $\text{SR}(\kappa, \lambda)$ is the statement that for every stationary subset $S$ of $\kappa \cap \text{Cof}(\lambda)$, there is a $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in $\gamma$.

We’ll stick to $\lambda = \omega$ for convenience.
Stationary reflection

Definition
For regular $\kappa > \lambda$, $\text{SR}(\kappa, \lambda)$ is the statement that for every stationary subset $S$ of $\kappa \cap \text{Cof}(\lambda)$, there is a $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in $\gamma$.

We’ll stick to $\lambda = \omega$ for convenience.

Proposition (Solovay, Reinhardt and Kanamori)

$$\text{SR}(\alpha^+, \lambda) \rightarrow \neg \square_{\alpha}$$
Stationary reflection

Definition
For regular \( \kappa > \lambda \), \( \text{SR}(\kappa, \lambda) \) is the statement that for every stationary subset \( S \) of \( \kappa \cap \text{Cof}(\lambda) \), there is a \( \gamma < \kappa \) such that \( S \cap \gamma \) is stationary in \( \gamma \).

We’ll stick to \( \lambda = \omega \) for convenience.

Proposition (Solovay, Reinhardt and Kanamori)

\[
\text{SR}(\alpha^+, \lambda) \rightarrow \neg \square_{\alpha}
\]

Proof.
Suppose \( C \) is a \( \square_{\alpha} \)-sequence; \( \gamma \mapsto \text{ot}(C_\gamma) \) is then a regressive function on \( (\alpha, \alpha^+) \cap \text{Cof}(\lambda) \). Thus, there is some stationary set \( S \) on which this function is constant, say with value \( \zeta \). If \( S \cap \gamma \) were stationary in \( \gamma \) for some \( \gamma < \kappa \), then \( \lim C_\gamma \cap S \) would be unbounded in \( \gamma \), and for \( \zeta \neq \xi \in \lim C_\gamma \cap S \), \( \text{ot}(C_\zeta) = \text{ot}(C_\gamma \cap \zeta) \neq \text{ot}(C_\gamma \cap \xi) = \text{ot}(C_\xi) \), contradicting the choice of \( S \). \( \square \)
\(\alpha^{++}\)-subcompactness implies \(SR(\alpha^+, \omega)\), but that’s too crude a tool for our purposes. All we really need beyond \(\alpha^+\)-subcompactness is for stationarity to be respected.
\( \alpha^{++} \)-subcompactness implies \( \text{SR}(\alpha^+, \omega) \), but that’s too crude a tool for our purposes. All we really need beyond \( \alpha^+ \)-subcompactness is for stationarity to be respected.

**Definition**

*For any cardinal \( \alpha \), we say that a cardinal \( \kappa \leq \alpha^+ \) is \((\alpha^+, \omega)\)-stationary subcompact if for every \( A \subseteq H_{\alpha^+} \) and every stationary set \( S \subseteq \alpha^+ \cap \text{Cof}(\omega) \), there exist \( \bar{\kappa} < \bar{\alpha}^+ < \kappa \), \( \bar{A} \subseteq H_{\bar{\alpha}^+} \), a stationary set \( \bar{S} \subseteq \bar{\alpha}^+ \cap \text{Cof}(\omega) \) and an elementary embedding

\[
\pi : (H_{\bar{\alpha}^+}, \in, \bar{A}, \bar{S}) \rightarrow (H_{\alpha^+}, \in, A, S)
\]

with critical point \( \bar{\kappa} \) such that \( \pi(\bar{\kappa}) = \kappa \).*

**Proposition**

*If some \( \kappa \leq \alpha \) is \((\alpha^+, \omega)\)-stationary subcompact, then \( \text{SR}(\alpha^+, \omega) \) holds.*
Again, we have a forcing reversal showing that this is optimal.

**Theorem (under GCH)**

Let $I$ be as defined above, and similarly let

$$J = \{ \alpha \mid \exists \kappa \leq \alpha (\kappa \text{ is } \alpha^+\text{-stationary subcompact}) \} \subseteq I.$$

Then there is a cofinality-preserving class forcing $\mathbb{P}$ such that for any $\mathbb{P}$-generic $G$ the following hold.

1. $\text{SR}(\alpha^+,\omega)$ fails in $V[G]$ for all $\alpha \notin J$.
2. $\Box_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.
3. If $\kappa \leq \alpha$ are such that $V \models \kappa$ is $(\alpha^+,\omega)$-stationary subcompact, then $V[G] \models \kappa$ is $(\alpha^+,\omega)$-stationary subcompact. In particular, $J^{V[G]} = J$.

Moreover, $\mathbb{P}$ preserves all $\omega$-superstrong cardinals.
Other weakenings of $\square_\alpha$

Schimmerling introduced the following hierarchy of weak squares.

**Definition**

*For any cardinal $\alpha$, a $\square_\alpha,<\mu$-sequence is a sequence $\langle C_\beta \mid \beta \in \alpha^+ \cap Lim \rangle$ such that for every $\beta \in \alpha^+ \cap Lim$,*

- $C_\beta$ is a set of closed unbounded subsets of $\beta$,
- $1 \leq |C_\beta| < \mu$,
- $\ot(C) \leq \alpha$ for every $C \in C_\beta$,
- for any $C \in C_\beta$ and $\gamma \in \lim(C)$, $C \cap \gamma \in C_\gamma$.

*We say $\square_\alpha,<\mu$ holds if there exists a $\square_\alpha,<\mu$-sequence, and we write $\square_\alpha,\nu$ for $\square_\alpha,<\nu^+$.*
Some of these weak forms of square are also precluded by \( \alpha^+ \)-subcompactness.
Some of these weak forms of square are also precluded by $\alpha^+-$subcompactness.

**Theorem**

*Suppose $\kappa$ is $\alpha^+$-subcompact for some $\kappa \leq \alpha$. Then $\Box_{\alpha,<\text{cf}(\alpha)}$ fails.*
Some of these weak forms of square are also precluded by $\alpha^+$-subcompactness.

**Theorem**

Suppose $\kappa$ is $\alpha^+$-subcompact for some $\kappa \leq \alpha$. Then $\square_{\alpha,<\text{cf}(\alpha)}$ fails.

Note that under the GCH, $\square_{\alpha,\alpha}$ holds for all regular cardinals.
Some of these weak forms of square are also precluded by $\alpha^+$-subcompactness.

**Theorem**

Suppose $\kappa$ is $\alpha^+$-subcompact for some $\kappa \leq \alpha$. Then $\square_{\alpha, < \text{cf}(\alpha)}$ fails.

Note that under the GCH, $\square_{\alpha, \alpha}$ holds for all regular cardinals.

**Theorem**

Suppose $\kappa$ is $\alpha^+$-subcompact for some $\kappa \leq \alpha$ with $\kappa > \text{cf}(\alpha)$. Then $\square_{\alpha, \alpha}$ fails.
Again, we show by forcing that this is all that can be said.

**Theorem (under GCH)**

As before, let

$$ I = \{ \alpha | \exists \kappa \leq \alpha ( \kappa \text{ is } \alpha^+-\text{subcompact}) \} , $$

and similarly let

$$ K = \{ \alpha | \exists \kappa > \text{cf}(\alpha) ( \kappa \text{ is } \alpha^+-\text{subcompact}) \} \subseteq I . $$

Then there is a cofinality-preserving partial order $\mathbb{P}$ such that for any $\mathbb{P}$-generic $G$ the following hold.

1. $\Box_\alpha$ holds in $V[G]$ for all $\alpha \notin I$.
2. $\Box_{\alpha, \text{cf}(\alpha)}$ holds in $V[G]$ for all $\alpha \notin K$.
3. $I^{V[G]} = I$.