The infinite random simplicial complex

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Abstract

We study the Fraïssé limit of the class of all finite simplicial complexes. Whilst the natural model-theoretic setting for this class uses an infinite language, a range of results associated with Fraïssé limits of structures for finite languages carry across to this important example. We introduce the notion of a local class, with the class of finite simplicial complexes as an archetypal example, and in this general context prove the existence of a 0-1 law and other basic model-theoretic results. Constraining to the case where all relations are symmetric, we show that every direct limit of finite groups, and every metrizable profinite group, appears as a subgroup of the automorphism group of the Fraïssé limit. Finally, for the specific case of simplicial complexes, we show that its geometric realisation is topologically surprisingly simple: despite the combinatorial complexity of the Fraïssé limit, its geometric realisation is homeomorphic to the infinite simplex.

1 Introduction

Simplicial complexes are important objects of study in a variety of areas of mathematics, including algebraic topology and combinatorics. Much at-
tention has been paid to random $d$-dimensional simplicial complexes with complete $(d-1)$-skeleton, for fixed $d$. Indeed, Rado’s original random graph paper [18] shows that there is a universal countable $d$-dimensional simplicial complex. Blass and Harary [1] have shown in this context that a 0-1 law holds, and more recently questions about homology of random $d$-dimensional simplicial complexes have come to the fore, such as in [11] and [14].

However, all of this leaves open the analogous questions for random simplicial complexes with no such dimension restriction. In the present paper we address this topic. Rather than the uniform probability measure on the set of $n$ vertex simplicial complexes, we consider what is arguably a more natural measure for this context: the measure induced by building up simplicial complexes inductively by dimension, tossing a fair coin for each potential $n$-simplex (with its full $(n-1)$-skeleton already in the complex) to determine whether it will be in the final simplicial complex. We show in Theorem 32 that with this probability measure the class of all finite simplicial complexes bears a 0-1 law.

The proof of Theorem 32 is actually a relatively simple modification of known techniques from the theory of Fraïssé classes, and it is in this setting that we frame our results. We define the notion of a local class, a notion which resembles and is closely related to Oberschelp’s notion of a parametric class [17], and which will include the class of finite simplicial complexes as an example. We then isolate a property common to local classes and parametric classes, the Adoptive Property, and use this property to demonstrate the existence of a 0-1 law for local classes. The Adoptive Property is also similar to but not quite subsumed by an extant notion in the literature, that of admitting substitutions due to Koponen [9], and the arguments used are correspondingly similar.

Integral to this approach is the countable structure known as the Fraïssé limit of the class of structures in question, which in the simplicial complex case we dub the infinite random simplicial complex $F_{SC}$. The automorphism groups of Fraïssé limits are an area of extensive research (see for example [12, 19, 20]), and as a nod to this we prove in Section 6 some basic results about the range of subgroups of $\text{Aut}(F_{SC})$. Again our results hold in broader generality, applying to the automorphism group of any local class for which all relations are symmetric.

Linial and Meshulam [11] conclude by opining that “...further study of topological properties of random complexes will prove both interesting and useful.” In Section 7 we prove a surprising result about the topology of $F_{SC}$:
despite its high combinatorial complexity, \( F_{SC} \) has a topologically very simple geometric realisation, homeomorphic to the infinite simplex. This naturally raises the question of whether properties such as contractibility are held by almost all finite simplicial complexes: a negative answer would yield a proof that the property in question is not first order definable.

2 Preliminaries and definitions

We begin by fixing our (mostly standard) model-theoretic notation; see for example [3, §1] or [8, §1] for further details. Let \( \Sigma = \{ R_n \mid n \in \mathbb{N} \} \) be a relational signature (so, each \( R_n \) is a relation symbol for our language). We write \( \Sigma \)-structures as \( M = \langle |M|, R^M_0, R^M_1, \ldots \rangle \), where \( |M| \) is the underlying set of \( M \) and \( R^M_n \) is the interpretation of \( R_n \) in \( M \): that is, if the arity of \( R_n \) is \( a_n \), then \( R^M_n \) is the set of \( a_n \)-tuples of elements of \( |M| \) for which the relation \( R_n \) is said to hold in \( M \). For a \( \Sigma \)-structure \( M \) and a first order sentence \( \varphi \) over \( \Sigma \) the notation \( M \models \varphi \) means that \( \varphi \) holds of \( M \); similarly, for a set \( \Phi \) of first order sentences, \( M \models \Phi \) means that every \( \varphi \in \Phi \) holds of \( M \). We use \( |M| \) to denote the cardinality of \( M \), that is, the cardinality of the underlying set \( |M| \) of \( M \). For any set \( X \) we denote by \([X]^m\) the set of \( m \)-element subsets of \( X \), and by \([X]^{fin}\) the set of finite subsets of \( X \). We use \( X \subset Y \) to denote that \( X \) is a (not necessarily proper) subset of \( Y \).

We shall be primarily interested in classes of finite structures (in particular, finite simplicial complexes) up to isomorphism; for this, it will be convenient to assume that these structures have subsets of \( \mathbb{N} \) as their underlying sets. Thus, by “class of structures” we shall mean a class \( K \) of finite structures \( M \) with \( |M| \subset \mathbb{N} \). Concomitantly, we use \( \text{Mod} \Phi \) to denote the class of finite \( \Sigma \)-structures \( M \) such that \( M \models \Phi \) and \( |M| \subset \mathbb{N} \), and similarly we denote by \( \text{Str} \Sigma \) the class of all finite \( \Sigma \)-structures \( M \) with \( |M| \subset \mathbb{N} \). Identity of formulas is denoted by \( \equiv \), not to be confused with \( = \) (which is used within formulas); thus for example, \( \varphi \equiv R(x_1, \ldots, x_n) \) means that \( \varphi \) is the formula \( R(x_1, \ldots, x_n) \).

It will be important to distinguish between the following two notions of “substructure”, as both have a role to play in what is to follow.

**Definition 1.** Suppose \( B \) is a structure for a relational signature \( \Sigma \). A substructure \( A \) of \( B \) is the induced substructure on a subset of \( B \). That is, \( A \) is a substructure of \( B \) if \( |A| \subset |B| \) and for every relation symbol \( R \) in \( \Sigma \) and tuple \( a \) from \( A \) of length the arity of \( R \), \( R(a) \) holds in \( A \) if and only if \( R(a) \)
holds in $B$. A subobject $B'$ of $B$ is a $\Sigma$-structure $B'$ such that $|B'| \subset |B|$ and the inclusion map is a $\Sigma$-homomorphism. That is, for every relation symbol $R$ in $\Sigma$ and tuple $b$ from $B'$ of length the arity of $R$, if $R(b)$ holds in $B'$ then $R(b)$ holds in $B$, but not necessarily conversely.

If $X \subset |B|$, we shall sometimes write $B \upharpoonright X$ to mean the substructure of $B$ with underlying set $X$. Koponen [9] refers to subobjects as *weak substructures*, but we shall stick with this less verbose terminology from category theory.

The main thrust of our results will be that techniques from the finite signature case can be made to work for suitably “locally finite” theories over infinite signatures. For this, the following two definitions will be crucial.

**Definition 2.** A relational signature $\Sigma$ is locally finite if for every $n \in \mathbb{N}$ there are only finitely many $n$-ary relation symbols in $\Sigma$.

Local finiteness of the signature will not suffice on its own, as high arity relations with repeated entries can emulate low arity relations. To overcome this, we employ the following.

**Definition 3.** For any signature $\Sigma$, we denote by $\Phi_{\Sigma}^{\text{GI}}$ the set of axioms fitting the following schema:

**General Irreflexivity (GI):** For all natural numbers $n \geq 2$, all $R$ in $\Sigma$ of arity $n$, and all distinct $i,j \in \{1, \ldots, n\}$,

$$\forall x_1 \cdots \forall x_n \left( R(x_1, \ldots, x_n) \rightarrow (x_i \neq x_j) \right).$$

Also important will be the following kind of subobject, as they capture local properties.

**Definition 4.** Let $\Sigma$ be a relational signature, let $A$ be an $\Sigma$-structure, and let $n \geq 1$ be a natural number. The $n$-frame $A^{(n)}$ of $A$ is the subobject of $A$ with the same underlying set $|A|$, the same interpretations $R^A$ of the $m$-ary relations $R$ in $\Sigma$ for all $m \leq n$, but the empty interpretation for all higher arity relations.

We put parentheses in the superscript for an $n$-frame to distinguish from the standard indexing convention for skeleta of simplicial complexes, and to emphasise the point of view of $A^{(n)}$ as a kind of $n$-th order approximation to $A$.

We immediately make two simple but important observations.
Lemma 5. If $K \subset \text{Mod}(\Phi_{\Sigma}^\text{GI})$ for a locally finite relational signature $\Sigma$, then for any $A \in K$ of cardinality $n \in \mathbb{N}$, $A$ is equal to its $n$-frame $A^{(n)}$. \hfill $\square$

Thus, the $n$-frame of a structure $B$ can be thought of as being built up in a natural way from the cardinality $n$ substructures of $B$.

Lemma 6. If $K \subset \text{Mod}(\Phi_{\Sigma}^\text{GI})$ for a locally finite relational signature $\Sigma$, then there are up to isomorphism only finitely many structures in $K$ of cardinality $n$. \hfill $\square$

2.1 Fraïssé limits

We recall the basic theory of Fraïssé limits; see for example [8, Chapter 7] for more details. Let $\Sigma$ be a countable signature (we shall give the explicit signature for simplicial complexes below). A countably infinite class $K$ of finitely generated $\Sigma$-structures is said to be a Fraïssé class if it satisfies the following three properties.

Hereditary Property, HP. For any $B \in K$ and any finitely generated substructure $A$ of $B$, $A$ is also in $K$.

Joint Embedding Property, JEP. For any $B, C \in K$, there is a $D \in K$ such that both $B$ and $C$ embed into $D$.

Amalgamation Property, AP. Suppose that $A, B, C \in K$ with embeddings $f_B : A \to B$ and $f_C : A \to C$. Then there are $D \in K$ and embeddings $g_B : B \to D$ and $g_C : C \to D$ such that $g_B \circ f_B = g_C \circ f_C$.

Note that here substructure means induced substructure, and an embedding is an injective homomorphism also preserving the negations of the relations (thus, an isomorphism to its image). Also note that for relational languages — that is, those without function or constant symbols — finitely generated simply means finite. For any Fraïssé class $K$, there is a countable $\Sigma$-structure $F$ such that $K$ is the class of all finitely generated $\Sigma$-structures (up to isomorphism) that can be embedded into $F$ ($K$ is the age of $F$) and $F$ has the following homogeneity property, which, following Hodges [8, Section 7.1], we refer to as weak homogeneity.

Definition 7. A $\Sigma$-structure $F$ is weakly homogeneous if for any finitely generated $\Sigma$-structure $B$ that embeds into $F$, any substructure $A$ of $B$, and any embedding $f$ of $A$ into $F$, there is an embedding $g$ of $B$ into $F$ extending $f$. 

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Weak homogeneity is sometimes also referred to as the extension property. Any two countable weakly homogeneous structures of the same age are isomorphic (see for example [8, Lemma 7.1.4]), so such a structure $F$ is unique up to isomorphism. This $F$ is known as the Fraïssé limit of $K$.

A well-known example of a Fraïssé limit is the countable random graph, or Rado graph, which is the Fraïssé limit of the class of finite graphs. However, there are many other kinds of structures to which the theory can be applied, such as finite groups (using Neumann’s permutation products [15] for AP; the Fraïssé limit group was studied by Hall [7]) and finite rational metric spaces (the completion of the Fraïssé limit of which is the well-known Urysohn space [21]).

The elements of a Fraïssé class clearly need only be given up to isomorphism. For concreteness, we shall always assume that any Fraïssé class $K$ only contains members $M$ with underlying set a subset of $\mathbb{N}$ (which we take to include $0$), and that $K$ is closed under isomorphism of such structures.

2.2 Simplicial complexes

Recall that a simplicial complex on a set $V$ is a subset $\Delta$ of $[V]^{\text{fin}}$ (the set of finite subsets or simplices of $V$) which is closed under inclusion: if $x$ is in $\Delta$ and $y \subseteq x$ then $y \in \Delta$. We refer to those simplices $x$ in $\Delta$ as faces of $\Delta$. To couch simplicial complexes in a model-theoretic framework, we make the following definitions.

Definition 8. The signature of simplicial complexes $\Sigma_{\text{sc}}$ is the set $\Sigma_{\text{sc}} = \{ S_i \mid i \in \mathbb{N} \}$, where for each $i \in \mathbb{N}$, $S_i$ is an $i+1$-ary relation symbol. The language of simplicial complexes $L_{\text{sc}}$ is the first order language comprising formulae built from $\Sigma_{\text{sc}}$ in the usual way.

Definition 9. A simplicial complex is formally a $\Sigma_{\text{sc}}$ structure satisfying $\Phi_{GI}$ and all axioms fitting either of the following two further schemata:

Symmetry: For every positive $n$ and every permutation $\sigma$ of the set $\{0, \ldots, n\}$,
\[
\forall x_0 \cdots \forall x_n \left( S_n(x_0, \ldots, x_n) \rightarrow S_n(x_{\sigma(0)}, \ldots, x_{\sigma(n)}) \right).
\]

Subset Closure: For every positive $n$,
\[
\forall x_0 \cdots \forall x_n (S_n(x_0, x_1, \ldots, x_n) \rightarrow S_{n-1}(x_0, \ldots, x_{n-1})).
\]
General Irreflexivity and Symmetry allow us to encode the \(n + 1\)-element sets of a subset of \(V^{\text{fin}}\) by (all of) the ordered \(n + 1\)-tuples of those elements, and then the Subset Closure axiom schema describes simplicial complexes in this context.

Clearly the class of finite simplicial complexes satisfies the Hereditary Property, the Amalgamation Property and the Joint Embedding Property. Thus, we may make the following definition.

**Definition 10.** The infinite random simplicial complex \(F_{\text{SC}}\) is the Fraïssé limit of the class of finite simplicial complexes.

The \(k\)-frame \(\Delta^{(k)}\) of a simplicial complex \(\Delta\) is an important notion in the study of simplicial complexes, called the \((k - 1)\)-skeleton of \(\Delta\) and traditionally denoted \(\Delta^{k-1}\). This shift of 1 from the cardinality of the faces to the index makes sense as the dimension of the geometric realisation of \(\Delta\) (see Section 7 below), but would make little sense in the general model-theoretic setting we use, so we have elected to introduce the new notation and terminology of \(k\)-frames.

## 3 Local classes

A major feature of the theory surrounding Fraïssé limits is the existence of 0-1 laws. Glebskii, Kogan, Liogon’kii and Talanov [6] and Fagin [5] showed that the Fraïssé class of all structures for a finite language bears a 0-1 law. Oberschelp [16] generalised this to so-called *parametric classes* (see below for a definition). More recently, Koponen [9] has generalised further, undertaking a detailed study of 0-1 laws based on extension axioms (which we too shall employ). A common feature of these results is that the underlying signature is generally taken to be finite (although Koponen’s framework in [9] frequently admits the infinite case). Indeed this is reasonable — in general, classes of structures for an infinite signature will have infinitely many structures of each finite cardinality, ruling out numerical asymptotic probability calculations.

However, we show below that such arguments can also be made to go through for simplicial complexes (with their infinite signature described in Section 2.2) and other similarly “local” classes for infinite languages. In this section we give a suitable definition of what it means to be a local class for this purpose.

For context, we begin by recalling Oberschelp’s notion of a parametric class.
Definition 11. Suppose $\Sigma$ is a relational signature. A first-order sentence $\varphi$ over $\Sigma$ is parametric if it is a conjunction of sentences of the form

$$\forall x_1 \cdots \forall x_m ( ( \bigwedge_{1 \leq i < j \leq m} x_i \neq x_j ) \rightarrow \psi )$$

where $m > 0$ and $\psi$ is a Boolean combination of terms $R(y_1, \ldots, y_n)$ with $R \in \Sigma$ and $\{y_1, \ldots, y_n\} = \{x_1, \ldots, x_m\}$. A class $K$ of structures is said to be parametric if $K = \text{Mod}(\varphi)$ for a parametric sentence $\varphi$.

Now to our variant.

Definition 12. Suppose $\Sigma$ is a relational signature. A first-order sentence $\varphi$ over $\Sigma$ is local if it is of the form

$$\forall x_1 \cdots \forall x_m (R(x_1, \ldots, x_m) \rightarrow \psi),$$

where $m > 0$ and $\psi$ is a quantifier-free formula.

Note in particular that since $\varphi$ is a sentence, the free variables of $\psi$ are among $\{x_1, \ldots, x_m\}$; however we make no assumption that they all appear in $\psi$.

Definition 13. Suppose $\Sigma$ is a relational language with finitely many relations of each arity. A class $K$ of $\Sigma$-structures is local if there is a set $\Phi \supset \Phi^\Sigma_{GI}$ of local sentences such that $K = \text{Mod} \Phi$.

Note that all of the axioms arising from our simplicial complex axiom schemata are local, so the class of simplicial complexes is local.

An immediate connection between these definitions is the following. The subformula $\psi$ from Definition 11 can frequently be written in the form $R(x_1, \ldots, x_m) \rightarrow \psi'$, as in Definition 13, in which case the antecedent $\bigwedge x_i \neq x_j$ appearing in Definition 11 is unnecessary if the structures satisfy General Irreflexivity. Thus, many natural examples of parametric classes are local.

On the other hand, there are parametric classes that are not local. For example, as noted in [3, Example 4.2.2 (c)],

$$\forall \text{ distinct } x_1, \ldots, x_k (R(x_1, \ldots, x_k) \land \neg R(x_1, \ldots, x_k))$$

for $k$-ary $R$ is a parametric sentence with no models of size $k$ or more. In contrast, every local class $K$ contains structures of every finite size: because of the form of local sentences, any finite set endowed with the trivial (empty) interpretation for every relation will be a member of $K$. 
**Proposition 14.** Any local class is a Fraïssé class.

**Proof.** Suppose $K$ is a local class. The fact that HP holds is immediate from the fact that local sentences are only universally quantified. Given $B$ and $C$ in $K$, $D$ as in JEP may be constructed as the disjoint union of $B$ and $C$, with the induced relations for tuples entirely from $|B|$ or from $|C|$, and the relations always failing for mixed tuples. Similarly, given $f_B : A \to B$ and $f_C : A \to C$ as in AP, we may take $D$ with underlying set $|B| \cup |C|/f_B(a) \sim f_C(a)$, and relations induced from $B$ or $C$ if a tuple is entirely contained in one (or both) of them, and not holding otherwise. □

**Lemma 15.** For any local class $K$, $A \in K$, and $n \in \mathbb{N}$, $A(n) \in K$.

**Proof.** This is immediate from the definition of local sentences. □

Central to our probabilistic approach will be the following property, which unites local and parametric classes.

**Adoptive Property, AdP.** Suppose $B \in K$, $A \in K$ is a substructure of $B$ of cardinality $n$, and $A' \in K$ is a structure on $|A|$ with the same $(n-1)$-frame $A'^{(n-1)}$ as $A$. Then there is a structure $B' \in K$ with underlying set $|B|$ such that $(B')^{(n-1)} = B^{(n-1)}$, and for every $n$-ary relation $R \in \mathcal{L}$,

$$R^{B'} = (R^B \smallsetminus |A|^n) \cup R^{A'}.$$ 

That is, $B' \models R(b_1, \ldots, b_n)$ if and only if either

1. one of the $b_i$’s is not in $A$ and $B \models R(b_1, \ldots, b_n)$, or
2. $\{b_1, \ldots, b_n\} = |A|$ and $A' \models R(b_1, \ldots, b_n)$.

The Adoptive Property is closely related to Koponen’s notion of admitting substitutions [9], with the difference being that the Adoptive Property takes a more frame-by-frame approach. As such, the Adoptive Property is more constrained in its antecedent for lower dimensions, requiring that $A^{(n-1)} = A'^{(n-1)}$, and more liberal in its consequent in higher dimensions: AdP only requires that some $B'$ with the desired $n$-frame exists, with no demands placed on higher-arity relations, whereas Koponen’s substitutions define $B'$ in all dimensions from $B$ and $A'$. For parametric classes we may encapsulate this greater generality as follows.

**Proposition 16.** Every parametric class $K$ enjoys the Adoptive Property.
Proof. Because of the distinctness requirement on the variables in parametric sentences, we may take \( R^B = (R^B \setminus |A|^k) \cup R^{A'} \) for every \( k \)-ary relation \( R \) in the signature and every \( k \in \mathbb{N} \) and clearly get another member of \( K \). □

The extra flexibility in higher dimensions for the Adoptive Property is crucial for the example of simplicial complexes: if one removes a face from a simplicial complex, one must also remove every higher dimensional face which contains it.

**Proposition 17.** Every local class \( K \) enjoys the Adoptive Property.

*Proof.* Let \( A, A', B \in K \) and \( n \in \mathbb{N} \) be as in the statement of the Adoptive Property. By Lemma 15 we may assume without loss of generality that \( B = B^{(n)} \). Then \( B' = B'^{(n)} \) constructed from \( B \) as per the Adoptive Property is also a member of \( K \). Indeed, consider any local sentence \( \phi \equiv \forall x_1 \cdots \forall x_m (R(x_1, \ldots, x_m) \rightarrow \psi) \) in \( \Phi_K \). If \( m > n \) then \( \phi \) holds vacuously in \( B' \), and if \( m < n \) then \( \phi \) holds in \( B' \) because it holds in \( B \). If \( m = n \), then if \( \{b_1, \ldots, b_n\} = |A| \),

\[
B' \models R(b_1, \ldots, b_n) \rightarrow \psi(b_1, \ldots, b_n)
\]

because \( A' \models R(b_1, \ldots, b_n) \rightarrow \psi(b_1, \ldots, b_n) \),

and if \( \{b_1, \ldots, b_n\} \neq |A| \),

\[
B' \models R(b_1, \ldots, b_n) \rightarrow \psi(b_1, \ldots, b_n)
\]

because \( B \models R(b_1, \ldots, b_n) \rightarrow \psi(b_1, \ldots, b_n) \).

□

The restriction of the cardinality of \( A \) to \( n \) does not make the Adoptive Property weaker than it would otherwise have been.

**Lemma 18.** The following are equivalent for a class \( K \) of structures over a locally finite relational signature.

1. The Adoptive Property.

2. For every \( B \in K \), \( A \in K \) a finite substructure of \( B \) of cardinality at least \( n \), and \( A' \in K \) a structure on \( |A| \) with the same \( (n - 1) \)-frame \( A^{(n-1)} \) as \( A \), there is a structure \( B' \in K \) with underlying set \( |B| \) such that \( (B')^{(n-1)} = B^{(n-1)} \) and for every \( n \)-ary relation \( R \in \mathcal{L} \), \( R^{B'} = (R^B \setminus |A|^n) \cup R^{A'} \).
Proof. (2) $\Rightarrow$ (1) is immediate. For the converse, let $B$, $A$ and $A'$ be as in (2), and let $B_0 = B$. Let $X_1, \ldots, X_{|A|\binom{n}{n}}$ be an enumeration of the subsets of $|A|$ of cardinality $n$. Define $B_i$ recursively in $i$, letting $B_i$ equal the structure $B'$ obtained from the Adoptive Property applied to $B_{i-1}$ and $A' \upharpoonright X_i$. Then $B_{\binom{|A|}{n}}$ satisfies the requirements for the structure $B'$ as in (2).

3.1 Examples of local classes

The class of simplicial complexes is not the only natural example of a local class.

Hypergraphs

The definition of a hypergraph varies from reference to reference; we shall take a hypergraph $H$ on a set $X$ to be simply a subset of $[X]^{\text{fin}} = \{Y \subset X \mid Y \text{ is finite}\}$. As for simplicial complexes, we refer to $Y \in H$ as a face of $H$.

In our model-theoretic setting, this definition is encapsulated as follows.

Definition 19. A hypergraph is formally a $\Sigma_{sc}$-structure satisfying the General Irreflexivity and Symmetry schemata of Definitions 3 and 9 above.

Thus, the difference from the definition of a simplicial complex is that we do not require that the Subset Closure axiom schema be satisfied by hypergraphs. Clearly the class of finite hypergraphs is a local class.

Note that if all of the faces of a hypergraph $H$ are of the same cardinality $d$, it may be identified with the simplicial complex with complete $(d - 1)$-frame. Indeed, Rado’s construction [18] of the Fraïssé limit of $d$-dimensional simplicial complexes with complete $(d - 1)$-skeleton, which was mentioned in the introduction, is actually cast in these terms.

Sperner families

Recall that a Sperner family $A$ on a set $X$ is an antichain in $[X]^{\text{fin}}$, that is a subset $A$ of $[X]^{\text{fin}}$ with the property that if $Y, Z \in A$ and $Y \subset Z$, then $Y = Z$. This too may be formalised to yield an example of a local class.

Definition 20. A Sperner family is a hypergraph satisfying the following further axiom schema.
Non-subset: For every $m < n$,

$$\forall x_0 \cdots \forall x_n (S_n(x_0, \ldots, x_n) \rightarrow \neg S_m(x_0, \ldots, x_m)).$$

Again, it is clear that Sperner families form a local class. It should be noted, on the other hand, that simplicial complexes are more closely related to Sperner families than just via our formalism: a simplicial complex may be identified with the Sperner family of its minimal non-faces (or for finite simplicial complexes, the Sperner family of its maximal faces).

4 First order theory

As mentioned above, with our definition of local classes in place, a range of model-theoretic consequences may be obtained by suitably modifying standard arguments.

Let us fix $\Sigma = \{R_{m,k} \mid 0 < m \in \mathbb{N}, 1 \leq k \leq k_m\}$, a locally finite relational signature, where each $R_{m,k}$ is an $m$-ary relation symbol, so that $k_m$ is the number of $m$-ary relations in $\Sigma$. Let $\mathbf{K} = \text{Mod}(\Phi_{\mathbf{K}})$ be a local class of finite $\Sigma$-structures, and let $F$ be the Fraïssé limit of $\mathbf{K}$. Of course, for concreteness the reader may think of our motivating example, with $\Sigma = \Sigma_{sc}$ and $\mathbf{K}$ the class of finite simplicial complexes. Let $\text{Th}_F$ denote the complete first order theory of $F$ over the signature $\Sigma$, that is, the set of all sentences over $\Sigma$ that are true in $F$. A variety of nice results are known regarding the theory of the Fraïssé limit of a Fraïssé class for a finite language; we show here that some of the most important results also hold when $\mathbf{K}$ is a local Fraïssé class.

**Theorem 21.** (i) Every countable model of $\text{Th}_F$ is isomorphic to $F$ (that is, $\text{Th}_F$ is countably categorical).

(ii) $\text{Th}_F$ satisfies quantifier elimination: for every formula $\varphi$ of $\mathcal{L}_{sc}$ with free variables $x_1, \ldots, x_n$, there is a formula $\psi$ involving no quantifiers such that

$$\forall x_1 \cdots \forall x_n \left( \varphi \iff \psi \right)$$

is in $\text{Th}_F$.

**Proof.** The proofs of these facts are much like the standard proofs for the finite $\Sigma$ case, as for example in [8, Theorem 7.4.1], but using the fact that $\mathbf{K}$ is local in place of finiteness.
We start by axiomatising $Th_F$. This will be by means of a formalisation of the weak homogeneity of $F$, as in Definition 7. By induction and using the Hereditary Property, it is sufficient to consider structures $A$ and $B$ in $K$ such that $B$ is a one point extension of $A$. Recall that we assume that every member of $K$ has underlying set a subset of $\mathbb{N}$, and that $K$ is assumed to be closed under isomorphism of such structures. Thus, we may assume without loss of generality that $|B| = \{1, \ldots, n\}$ and $A$ is the restriction of $B$ to $\{1, \ldots, n-1\}$. For every $m \leq n$, every $m$-ary relation symbol $R_{m,k} \in \Sigma$, and all $i_1, \ldots, i_m$ between 1 and $n$ inclusive, let $\bar{R}_k^{B;i_1,\ldots,i_m}$ denote the relation symbol $R_{m,k}$ if $R_{m,k}(i_1,\ldots,i_m)$ holds in $B$, and $\neg R_{m,k}$ if not. Now let $\varphi_B$ denote the extension axiom corresponding to $A \hookrightarrow B$, that is, the formal sentence that can be written as

$$\forall x_1 \cdots \forall x_{n-1} \left( \bigwedge_{1 \leq i \neq j \leq n-1} x_i \neq x_j \land \bigwedge_{m=1}^{n-1} \bigwedge_{k=1}^{k_m} \bigwedge_{i_1=1}^{i_m} \bar{R}_{m,k}^{B;i_1,\ldots,i_m}(x_{i_1},\ldots,x_{i_m}) \right) \to \exists x_n \left( \bigwedge_{i=1}^{n-1} x_i \neq x_n \land \bigwedge_{m=1}^n \bigwedge_{k=1}^{k_m} \bigwedge_{i_1=1}^{i_m} \bar{R}_{m,k}^{B;i_1,\ldots,i_m}(x_{i_1},\ldots,x_{i_m}) \right).$$

Thus, $\varphi_B$ formally expresses the statement that if there is a substructure isomorphic to $A$, then there is a substructure extending it by one vertex which is isomorphic to $B$ (we of course take empty conjunctions to be true, so that in the $n = 1$ case $\varphi_B$ reduces to a simple statement expressing the existence of a substructure isomorphic to $B$). These formulas $\varphi_B$ may be thought of as generalisations of the well-known property of the infinite random graph, that given any two finite sets of vertices, there is a vertex adjacent to every vertex in the first set and not adjacent to any vertex in the second set. In the random graph case the structure of $A$ (that is, the induced subgraph on the union of the two sets) is irrelevant, but in our setting this will not in general be the case.

Note that in stating $\varphi_B$ we have made crucial use of the fact that $K$ is local. In particular, the conjunction over $k$ is finite for each $m$ because $K$ is locally finite, and the fact that we are free to ignore higher-arity relations follows from General Irreflexivity.

The sentences $\varphi_B$ encapsulate the weak homogeneity of $F$; moreover by induction they show that $K$ is a subset of the age of any structure that
satisfies all of them. Thus, any structure satisfying
\[ \Phi_F = \Phi_K \cup \left\{ \varphi_B \mid B \in K \land \exists n \in \mathbb{N} \left( |B| = \{1, \ldots, n\} \right) \right\} \]
has age equal to $K$. Clearly $F \models \Phi_K$: any (local) sentence $\varphi \in \Phi_K$ has negation of the form $\exists x_1 \cdots \exists x_m(\psi)$ where $\psi$ is quantifier-free, so a witnessing tuple in $F$ would also witness the failure of $\varphi$ in a finite substructure. Thus, $\Phi_F$ is a subset of $Th_F$, such that that any model of $\Phi_F$ is a weakly homogeneous structure with age $K$, and therefore is isomorphic to $F$. So $Th_F$ is countably categorical, and by the Gödel completeness theorem we have that $\Phi_F$ provides a complete axiomatisation of $Th_F$.

(ii) Suppose $\varphi$ is a formula over $\Sigma$ with free variables $x_1, \ldots, x_n$. We shall exhibit a formula $\psi$ with free variables $x_1, \ldots, x_n$ and no quantifiers, such that relative to $Th_F$, $\psi$ is equivalent to $\varphi$. If there is no tuple $r = (r_1, \ldots, r_n)$ from $F$ such that $\varphi(r)$ holds in $F$, then any tautologically false formula such as $\neg(x_1 = x_1)$ will do for $\psi$. Otherwise, consider any tuple $r = (r_1, \ldots, r_n)$ from $F$ such that $\varphi(r)$ holds in $F$. A standard property of Fraïssé limits is that they are ultrahomogeneous: any isomorphism of finite substructures of $F$ extends to an automorphism of $F$ (see for example Hodges [8, Lemma 7.1.4]). Thus, for any $r'$ with a coordinate-respecting isomorphism $f: F \rightarrow F$ from $F \upharpoonright r$ to $F \upharpoonright r'$, we have an automorphism $\bar{f}$ of $F$ extending $f$. Since automorphisms preserve all formulas, $\varphi(r')$ will therefore also hold in $F$. Hence, whether $\varphi$ holds on a given $r$ depends entirely on the induced substructure on the elements of $F$ appearing in $r$. So consider the set

\[ S_\varphi = \left\{ (A, a) \mid \begin{array}{l}
A \text{ is a } \Sigma\text{-structure on } \{1, \ldots, n\} \text{ in } K \\
\land a = (a_1, \ldots, a_n) \in \{1, \ldots, n\}^n \\
\land \text{there is an embedding } f: A \hookrightarrow F \text{ such that } \\
\varphi(f(a_1), \ldots, f(a_n))
\end{array} \right\} \]

(there are of course redundancies in $S_\varphi$ that could be eliminated, but it does not seem worth the notational hassle). Then $S_\varphi$ is finite since $K$ is local, and we may take our $\psi$ to be the formula

\[ \bigvee_{(A, a) \in S_\varphi} \bigwedge_{m=1}^{n} \bigwedge_{k=1}^{k_m} \bigwedge_{1 \leq i_1 < \cdots < i_m \leq n} \bar{R}_{m, k}^{A; a_1, \ldots, a_m}(x_{i_1}, \ldots, x_{i_m}) \]

(with $\bar{R}_{m, k}^{A; a_1, \ldots, a_m}$ as in part (i)). This $\psi$ is a quantifier-free equivalent of $\varphi$ relative to $Th_F$. \qed
5 A probabilistic approach

In this section we shall consider a probabilistic characterisation of the Fraïssé limit of a local class $K$. This will give rise to our 0-1 law for first order sentences about elements of $K$, relative to the appropriate measure on the set of members of $K$ whose underlying set has $m$ elements. We shall start by discussing the case of simplicial complexes in order to convey the main ideas more concretely, and then move on to the general case of a local Fraïssé class $K$.

Probably the best-known description of the random graph is as the graph almost surely obtained (up to isomorphism) by including each possible edge with probability one half. Our generalisation of this to the simplicial complex context is as follows. Having constructed by tossing a fair coin a graph (in the infinite case, almost surely the random graph) as the 1-skeleton of a simplicial complex, we continue to make all decisions in the construction of the complex through all higher dimensions by tossing a fair coin. That is, given a triple $v_0, v_1, v_2$ of vertices such that $\{v_0, v_1\}, \{v_0, v_2\}$ and $\{v_1, v_2\}$ are all edges in the complex, we toss a fair coin to determine whether $\{v_0, v_1, v_2\}$ is a 2-face in the simplicial complex; and so on through all dimensions. Of course, to see whether a decision needs to be made for a given $n$-tuple of vertices, one only needs to know the decisions for subsets of those vertices. Despite the fact that the resulting probability measure for simplicial complexes seems very natural, it appears not to have been considered in detail before, although the version for finite simplicial complexes has been proposed [13] for use in social aggregation modelling, and for $d$-dimensional simplicial complexes for fixed $d$ this can be seen as a case of Koponen’s uniformly $(C_0, \ldots, C_{d+1})$-conditional probability measure [9, Definition 6.2].

We now work towards a formalisation of this for local classes in general. Let $K = \text{Mod}(\Phi_K)$ be a local class for a locally finite relational signature $\Sigma$. We continue in our assumption that each member $A$ of $K$ has underlying set $|A| \subset \mathbb{N}$ and $K$ is closed under isomorphism of such structures; recall also that for any structure $S$ we denote by $S^{(k)}$ the $k$-frame of $S$.

**Definition 22.** For $i \in \mathbb{N}$, we denote by $K^\infty$ the set of $\Sigma$-structures $M$ with underlying set $\mathbb{N}$ such that $M \models \Phi_K$. Similarly, we denote by $K^i$ the set of members of $K$ with underlying set $\{0, \ldots, i - 1\}$, we let $K^{<i}$ denote the set of members of $K$ with underlying set a subset of $\{0, \ldots, i - 1\}$, and for notational convenience we set $K^{<\infty} = K$. 

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The last part of this definition is apropos because we have the following equivalent characterisation of $K^\infty$.

**Lemma 23.** Let $M$ be a $\Sigma$-structure with $|M| = N$. Then $M$ is in $K^\infty$ if and only if every finite substructure of $M$ is in $K$.

**Proof.** This is immediate from the fact that local sentences are only universally quantified. \hfill \Box

For $i \in \mathbb{N} \cup \{\infty\}$ we define a topology on $K^i$ having as a base of open sets the collection of sets of the form

$$O_i(A, k) = \{S \in K^i \mid (S \upharpoonright |A|)^{(k)} = A^{(k)}\}$$

for $A \in K^{\leq i}$ and $k \in \mathbb{N}$.

If $i$ is a natural number, then the topology on $K^i$ defined by this base is discrete by Lemma 5. Moreover, there is a natural map $K^i \to K^{i-1}$ restricting a structure on $\{0, \ldots, i-1\}$ to the induced substructure on $\{0, \ldots, i-2\}$. The space $K^\infty$ is the inverse limit along these maps of the spaces $K^i$ as $i$ varies, and thus is profinite.

Note that $K^\infty$ with the above topology is compact: indeed, it can be viewed as a closed subset of the compact (using Lemma 6) Hausdorff space $\prod_{i \in \mathbb{N}} K^i$. Furthermore, every open cover of $K^\infty$ has a finite subcover with a finite disjoint refinement. This allows us to simply check finite additivity in order to apply Carathéodory's Extension Theorem.

Let $N(S, k)$ denote the number of distinct $(k+1)$-frames of structures $T \in K$ on the underlying set $|S|$ of $S$ such that $T^{(k)} = S^{(k)}$.

**Definition 24.** For all $i \in \mathbb{N} \cup \{\infty\}$, define $\mu_i$ on the base sets $O_i(A, k)$ ($A \in K^{\leq i}$) recursively in $k$ by setting $\mu_i(O_i(A, 0)) = 1$ and

$$\mu_i(O_i(A, k + 1)) = \frac{1}{N(A, k)} \mu_i(O_i(A, k)).$$

Observe that if $i \leq j$ and $A \in K^{\leq i} \subset K^{\leq j}$, then $\mu_i(O_i(A, k)) = \mu_j(O_j(A, k))$, and that for all $A \in K^{\leq i}$ and $k \in \mathbb{N}$, $\mu_i(O_i(A, k)) \neq 0$.

There are two parameters to our base sets: the substructure $A$ and the frame level $k$. Definition 24 is such that $\mu_i$ is clearly additive as one changes $k$, but we need to check that it is also additive for varying $A$. We introduce some notation to help with the proof of this fact.
**Definition 25.** For $X$ a finite subset of $\mathbb{N}$, $A$ an $\mathcal{L}$-structure with underlying set $|A| \subset X$, and $k \leq l$ natural numbers such that $k \leq \|A\|$ and $l \leq \|X\|$, define

$$K_{A,k}^{X,l} = \{ B^{(l)} | B \in K \land |B| = X \land B^{(k)} \upharpoonright |A| = A^{(k)} \}. $$

For example, $N(S, k) = \|K_{S,k}^{S,k+1}\|$.

**Lemma 26.** Suppose $S \in K$, $A$ is a substructure of $S$, and $1 \leq k \leq \|A\|$. Then for all $A' \in K_{A,k-1}^{\|A\|}$,

$$\|K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}\| = \|K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}\|. $$

In particular,

$$\|K_{S,k-1}^{S,k}\| = \|K_{A,k}^{\|A\|}\| \|K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}\| = N(A, k-1) \|K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}\|. $$

**Proof.** Lemma 18 provides a natural bijection between $K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}$ and $K_{S,k-1}^{S,k} \cap K_{A,k}^{A,k}$.

**Lemma 27.** Suppose $A \in K^{i}$, $0 \leq k \leq \|A\|$, and $X$ is a finite subset of $\{ j \in \mathbb{N} | j < i \}$ such that $|A| \subset X$. Then

$$\mu_i(O_i(A,k)) = \sum_{S \in K_{A,k}^{X,k}} \mu_i(O_i(S,k))$$

**Proof.** The proof is by induction on $k$, using the Adoptive Property in the form of Lemma 26. For $k = 0$ the statement holds because there is a unique 0-frame on any given underlying set: the structure with the empty interpretation for every relation.

So suppose the statement is true for values up to and including $k - 1$. Hence,

$$\mu_i(O_i(A,k-1)) = \sum_{S \in K_{A,k-1}^{X,k-1}} \mu_i(O_i(S,k-1))$$

$$N(A, k-1) \mu_i(O_i(A,k)) = \sum_{S \in K_{A,k-1}^{X,k-1}} N(S, k-1) \mu_i(O_i(S,k)),$$
where for every $S \in K_{A,k-1}^{X,k-1}$, $\bar{S}$ is a $k$-frame of the form $B^{(k)}$ for some $B \in K$ with $B^{(k-1)} = S$; by Definition 24, the expression is independent of the choice of $\bar{S}$. We have

\[ N(A, k - 1) \mu_i(O_i(A, k)) = \sum_{S \in K_{A,k-1}^{X,k-1}} \|K_{S,k-1}^X\| \mu_i(O_i(\bar{S}, k)) \]

\[ \mu_i(O_i(A, k)) = \sum_{S \in K_{A,k-1}^{X,k-1}} \|K_{S,k-1}^X \cap K_{A,k}^X\| \mu_i(O_i(S, k)) \]

\[ = \sum_{\bar{S} \in K_{A,k}^{X,k}} \mu(O_i(\bar{S})). \]

This result easily yields finite additivity for $\mu_i$ on the ring of sets generated by the base sets $O_i(A, k)$. Since all of the sets in this ring are clopen, this is equivalent to $\sigma$-additivity, and so applying the Carathéodory Extension Theorem in the $i = \infty$ case we may make the following definition.

**Definition 28.** For all $i \in \mathbb{N} \cup \{\infty\}$, the frame-wise uniform measure $\mu_i$ on $K^i$ is the probability measure induced by Definition 24.

The Adoptive Property tells us that whether a certain $n$-frame can occur on a given subset $X$ of a structure depends only on the $(n - 1)$-frame on $X$, and is independent of “what happens elsewhere”. With the frame-wise uniform measure, we also have this independence in the probability theory sense of the word. The following Lemma is indicative of this.

**Lemma 29.** For and $B \in K$ and any $k < \|B\|$,

\[ N(B, k) = \prod_{X \subset |B|, \|X\| = k+1} N(B \upharpoonright X, k). \]

**Proof.** The $(k+1)$-frame of a structure is of course determined by the $(k+1)$-frame of each of its $(k+1)$-element substructures. By the Hereditary Property, the right hand side of the equation is therefore the maximum possible value for $N(B, k)$, and the Adoptive Property shows that indeed every possibility occurs in $K$. \qed
Lemma 30. Suppose \( A, B \in K \), \( k \leq \|A\| \), \( l \leq \|B\| \), let \( Y = |A| \cap |B| \) and \( m = \min(l, \|Y\|) \), and suppose \( k \geq m \) and \( (A |Y)^{(m)} = (B |Y)^{(m)} \). Then
\[
\frac{\mu_\infty(O_\infty(A, k))}{\mu_\infty(O_\infty(A | Y, m))} = \frac{\mu_\infty(O_\infty(A, k) \cap O_\infty(B, l))}{\mu_\infty(O_\infty(B, l))}.
\]

More generally, suppose further that \( C_i \in K \) and \( n_i \in \mathbb{N} \) for \( i \) in a finite set \( I \) are such that \( n_i \leq \|C_i\| \), \( |A| \cap |C_i| = \emptyset \) for all \( i \in I \),
\[
(B \upharpoonright |B| \cap |C_i|)^{(\min(l, n_i, \|B| \cap |C_i|))} = (C_i \upharpoonright |B| \cap |C_i|)^{(\min(l, n_i, \|B| \cap |C_i|))}
\]
for all \( i \in I \), and
\[
(C_i \upharpoonright |C_i| \cap |C_j|)^{(\min(l, n_i, \|C_i| \cap |C_j|))} = (C_j \upharpoonright |C_i| \cap |C_j|)^{(\min(l, n_i, \|C_i| \cap |C_j|))}
\]
for all \( i \in I \). Let \( N \) be sufficiently large that \( A, B \) and all \( C_i \) are in \( K^{<N} \). Then
\[
\frac{\mu_N(O_N(A, k))}{\mu_N(O_N(A \upharpoonright Y, m))} = \frac{\mu_N(O_N(A, k) \cap O_N(B, l) \cap \bigcap_{i \in I} O_N(C_i, n_i))}{\mu_N(O_N(B, l) \cap \bigcap_{i \in I} O_N(C_i, n_i))}.
\]

Proof. This is clear by induction on \( k \), starting from 0 if \( |A| \setminus Y \) is nonempty and \( m \) otherwise. \( \square \)

Erdős and Renyi [4] showed that almost every countably infinite graph is isomorphic to the infinite random graph. With our frame-wise uniform measure \( \mu_\infty \) we have the analogous result for members of \( K^\infty \).

Theorem 31. Under the frame-wise uniform measure, almost every structure in \( K^\infty \) is isomorphic to the Fraïssé limit \( F \) of \( K \). That is,
\[
\mu_\infty(\{S \in K^\infty \mid S \cong F\}) = 1.
\]

Proof. The countable categoricity of \( Th_F \) given by Theorem 21 tells us that a countable \( \Sigma \)-structure is isomorphic to \( F \) if and only if it satisfies the axioms \( \Phi_F \) for \( Th_F \). By definition the elements of \( K^\infty \) satisfy the axioms \( \Phi_K \), so it suffices to show that almost every member of \( K^\infty \) satisfies all of the extension axioms \( \varphi_B \) as \( B \) varies over members of \( K \). There are only countably many such \( B \), so since \( \mu_\infty \) is countably additive, it suffices to show that for each such \( B \),
\[
\mu_\infty(\{S \in K^\infty \mid S \not\cong \varphi_B\}) = 0.
\]
So suppose $B$ is a member of $K$ with underlying set $|B| = \{0, \ldots, n\}$, and $A$ is the substructure induced on $\{0, \ldots, n - 1\}$. For each $n$-tuple $j = (j_0, \ldots, j_{n-1}) \in \mathbb{N}^n$ with distinct elements, let $K_{A,j}^\infty$ denote the set of members of $K^\infty$ into which $A$ embeds by the map $i \mapsto j_i$, that is, in the notation of Theorem 21,

$$K_{A,j}^\infty = \left\{ S \in K^\infty \mid \bigwedge_{m=1}^{n-1} \bigwedge_{k=1}^{k_m} \bigwedge_{0 \leq i_1, \ldots, i_m \leq n-1} \bar{R}_{m,k}^{A,i_1,\ldots,i_m}(j_{i_1}, \ldots, j_{i_m}) \right\}.$$ 

By $\sigma$-additivity again, it suffices to show that for every $j = (j_0, \ldots, j_{n-1}) \in \mathbb{N}^n$ with distinct elements,

$$\mu(\{ S \in K_{A,j}^\infty \mid \forall x_n \left( \bigvee_{i=0}^{n-1} j_i = x_n \lor \neg \bigwedge_{d=0}^{n-1} \bigwedge_{i_0, \ldots, i_{d-1} < n} \bar{R}_{d}^{B,i_0,\ldots,i_{d-1},n}(j_{i_0}, \ldots, j_{i_{d-1}}, x_n) \right) \}) = 0,$$

that is, the probability that no one-point extension of the image of $A$ in the structure is isomorphic to $B$ is 0. But now for members of $K_{A,j}^\infty$, the probability that the substructure induced on $j \cup \{j_n\}$ is isomorphic to $B$ for a given $j_n$ not in $j$ is non-zero, and takes the same value for all such $j_n$. Moreover, for a given $j_n$ it is independent of whether it holds for other elements of $\mathbb{N}$ not in $j$, by Lemma 30. Hence, the above measure is indeed 0, and we are done.

## 5.1 0-1 Law

In this section we show that, using the frame-wise uniform measure, there is a 0-1 law for arbitrary simplicial complexes, and likewise for the members of any local class. As mentioned above, Blass and Harary [1] have shown that there is a 0-1 law for simplicial complexes of dimension at most some given bound $d$, using for each $n$ the usual uniform measure on the set of at most $d$-dimensional simplicial complexes on $n$ vertices (that is, simply counting the simplicial complexes, with no regard for their structure). The restriction on the dimension that they impose is crucial for making sense of their 0-1 law — in their measure, almost every simplicial complex has a complete $(d - 1)$-skeleton. The 0-1 law we obtain below for the frame-wise
uniform measure imposes no such restriction, and indeed, in our 0-1 law the probability that even the underlying graph is complete converges to 0. In fact, in the frame-wise uniform measure, the underlying graph converges to the countable random graph.

Interestingly, in obtaining their 0-1 law Blass and Harary use axioms that generalise the extension axioms for the random graph, as our axioms $\varphi_B$ also do. However, their generalisation is more direct, involving a one-point extension in terms of two sets for which simplices should or should not be added. The simplicity of this generalisation is what forces the $(d - 1)$-skeleton to be complete, but of course it turns out that this is the right thing to do for the uniform measure. Our axioms $\varphi_B$ reflect more of a Fraïssé limit perspective, with the result that the 0-1 law that we obtain is for the frame-wise uniform measure. As such, our result is in a sense closer to the original 0-1 laws of Glebskii, Kogan, Liogon’kii and Talanov [6] and Fagin [5] than it is to the 0-1 law of Blass and Harary.

We continue to assume we have a fixed local class $K$ over a locally finite relational signature $\Sigma$ with Fraïssé limit $F$, and take $\mu_N$ to be the frame-wise uniform measure on $K^N$ for $N \in \mathbb{N} \cup \{\infty\}$, as defined above.

**Theorem 32.** Let $\varphi$ be a first-order sentence over $\Sigma$. As $N$ goes to infinity, $\mu_N(\{S \in K^N \mid S \models \varphi\})$ converges, with limiting value 1 if $\varphi$ holds of $F$, and 0 if not.

*Proof.* Since $Th_F$ is complete and axiomatised by $\Phi_F$ as in the proof of Theorem 21, every sentence $\psi$ in the language $\mathcal{L}$ will be provable or disprovable from $\Phi_F$. Of course any proof (of either $\psi$ or $\neg \psi$) will only involve finitely many axioms from $\Phi_F$. Therefore, if each axiom in $\Phi_F$ holds in members of $K^N$ with probability approaching 1, then every sentence $\psi$ true in $F$ will hold with probability approaching 1, and the negations of such statements will hold with probability approaching 0. We thus restrict attention to sentences in $\Phi_F$.

The idea of the proof is as for Theorem 31, in that as $N$ goes to infinity the probability that a given embedding of some $A$ does not extend to an embedding of $B$ drops to 0. However, in this finite case we must work harder to account for potential new embeddings of $A$. We use a proof reminiscent of Fagin [5, Theorem 2].

Suppose $A \in K$ has cardinality $n - 1$ and $B \in K$ is a one-point extension of $A$, and as in the proof of Theorem 21, let $\varphi_B$ denote the corresponding
extension axiom. Let \( \psi_B(x_1, \ldots, x_{n-1}) \) denote

\[
\bigwedge_{m=1}^{n-1} \bigwedge_{k=1}^{k_m} \bigwedge_{1 \leq i_1, \ldots, i_m < n} \bar{R}^{B;i_1,\ldots,i_m}(x_{i_1}, \ldots, x_{i_m}) \\
\bigwedge \forall x_n \left( \bigvee_{i=1}^{n-1} x_i = x_n \bigvee \bigwedge_{m=1}^{k_m} \bigvee_{1 \leq i_1, \ldots, i_m \leq n} \neg \bar{R}^{B;i_1,\ldots,i_m}(x_{i_1}, \ldots, x_{i_m}) \right),
\]

that is, the negation of the \( \varphi_B \) without the quantification or distinctness requirement on the variables \( x_1, \ldots, x_{n-1} \). Then for any \( N \) we have

\[
\mu_N(\{S \in K^N \mid S \not\models \varphi_B\}) \\
= \mu_N(\{S \in K^N \mid S \models \exists x_1 \cdots \exists x_{n-1} \left( \bigwedge_{1 \leq i \neq j \leq n-1} x_i \neq x_j \land \psi_B(x_1, \ldots, x_{n-1}) \right) \}) \\
\leq \sum \left\{ \mu_N(\{S \in K^N \mid S \models \psi_B(a_1, \ldots, a_{n-1})\}) \mid 0 \leq a_1, \ldots, a_{n-1} \leq N-1 \right. \\
\left. \land a_i \neq a_j \text{ for } i \neq j \right\} \\
= \binom{N}{n-1} \mu_N(\{S \in K^N \mid S \models \psi_B(1, \ldots, n-1)\}) \quad \text{by symmetry} \\
\leq N^{n-1} \mu_N(\{S \in K^N \mid S \models \psi_B(1, \ldots, n-1)\}) \\
= N^{n-1} \mu_N(\{S \in K^N \mid \forall y(\bigwedge_{i=1}^{n-1} y \neq i \rightarrow \bigvee_{m=1}^{n-1} \bigvee_{k=1}^{k_m} \bigvee_{1 \leq i_1, \ldots, i_m-1 \leq n} \neg \bar{R}^{B;i_1,\ldots,i_m-1,n}(x_{i_1}, \ldots, x_{i_m-1}, y))\})
\]

Let \( p = \mu_N(B, \|B\|)/\mu_N(A, \|A\|) \). Then applying Lemma 30, the last line above becomes \( N^{n-1} \mu_N(O_N(A, \|A\|))(1 - p)^{N-n+1} \). Since \( p \) is non-zero, this converges to 0 as \( N \) goes to infinity, so \( \lim_{N \to \infty} \mu_N(\{S \in K^N \mid S \not\models \varphi_B\}) = 0 \), as required.

We shall follow the convention that “almost all simplicial complexes satisfy \( \psi \)” is a shorthand for “as \( N \) goes to infinity, the measure of the set of simplicial complexes satisfying \( \psi \) goes to 1”.

Let us consider the particular case of simplicial complexes. The way we have set up our formalism, given an underlying set \( V \), only a subset of \( V \)
will be vertices (that is, singletons, or 0-dimensional faces) in our simplicial complex. In some contexts this is appropriate, but to fit better with conventions in the area, it is preferable to have all members of \( V \) as vertices in any simplicial complex “on \( V \)”. This is actually easy to achieve: we simply omit the relation \( S_0 \) from our signature, and otherwise proceed unchanged. Unless otherwise stated, this will be our approach henceforth.

It is obvious that our 0-1 law differs from that of Blass and Harary; for example, for the the frame-wise uniform measure, almost all simplicial complexes have an incomplete underlying graph. Indeed, a big advantage of the frame-wise uniform measure is that it respects the probabilities of underlying graphs, with the result that we can draw on the large body of knowledge regarding the random graph. For example, the result of Erdős and Renyi [4] that almost every graph is rigid lifts to give us the following.

**Proposition 33.** Under the frame-wise uniform measure, almost all simplicial complexes have trivial automorphism group.

**Proof.** Any automorphism of a simplicial complex is also an automorphism on the underlying graph. But now

\[
\mu_N(\{S \in K^N_{SC} \mid S^1 \text{ has trivial automorphism group}\})
\]

equals (by definition) the proportion of graphs on \( N \) vertices with trivial automorphism group, so

\[
\mu_N(\{S \in K^N_{SC} \mid S \text{ has trivial automorphism group}\}) \\
\geq \mu_N(\{S \in K^N_{SC} \mid S^1 \text{ has trivial automorphism group}\})
\]

which converges to 1 as \( N \) goes to infinity. \( \Box \)

We note that Bollobás and Palmer [2] obtained the same result for the usual uniform measure on \( K^N_{SC} \). Observe that this result also gives us a 0-1 law for an unlabelled form of the frame-wise uniform measure, with the infinite random simplicial complex \( F_{SC} \) still acting as the oracle for truth.

## 6 Algebraic properties

As mentioned in the introduction, the automorphism groups of homogeneous structures is an important area of current research. In this section we present
some basic results about the automorphism groups of $F_{SC}$ and similar Fraïssé limits, making use of locality. For this purpose, the symmetry properties common to simplicial complexes and the other examples in Section 3.1 are of course very relevant. Thus, for this section, let $K$ be a local class every member of which satisfies the Symmetry schema of Definition 9. Let $F$ denote the Fraïssé limit of $K$, and let $V = |F|$; in particular, we drop our assumption from the previous section that countable structures have underlying set $\mathbb{N}$ (although it would do no harm).

We recall some basic definitions from the theory of group actions, particularly profinite group actions — see for example [10] or [22] for background. Given an action of a group $G$ with identity element $e$ on a set $X$, for each $x \in X$ the stabilizer of $x$ is the subgroup of $g \in G$ such that $gx = x$. We thus say that an action has trivial stabilizers on $Y \subset X$ if for all $y \in Y$, $gy = y \rightarrow g = e$. An action is faithful if for all $g \neq h \in G$ there is an $x \in X$ such that $gx \neq hx$.

**Lemma 34.** Let $H$ be a finite group, $S \subset V$ a finite subset, and $v \in V \setminus S$ an element. Suppose that $\rho : H \times (F \upharpoonright S) \to F \upharpoonright S$ is a left action. Then there is a finite subset $S' \subset V$ such that $S \cup \{v\} \subset S'$ and the action $\rho$ extends to an action $\bar{\rho}$ of $H$ on $F \upharpoonright S'$ with trivial stabilizers on $S' \setminus S$.

**Proof.** Enumerate the group $H$ as $h_0 = e, h_1, \ldots, h_n$. We choose elements $v_{h_i} \in V \setminus S$ so that $\rho$ extends to an action $\bar{\rho}$ defined on $S' = S \cup \{v_h \mid h \in H\}$ by $\bar{\rho}(h_i, v_{h_j}) = v_{h_i h_j}$ and $\bar{\rho}(h, s) = \rho(h, s)$ for $s \in S$. We do this recursively in $i \leq n$. For $i = 0$, we choose $v_e = v$. For $i > 0$, let $H_i = \{h_0, \ldots, h_i\}$; we choose the vertex $v_{h_i}$ such that the following properties hold.

1. For all $j < i$, $v_{h_i}$ is different from $v_{h_j}$.

2. Suppose that $R$ is an $(m_1 + m_2)$-ary relation in the signature $\Sigma$ of $K$, $s$ is an $m_1$-tuple from $S$, and $h$ is an $m_2$-tuple from $H_i$. Let $v$ be the $m_2$-tuple $(v_g)_{g \in h}$, and suppose $h \in H$ is such that for every component $g$ of $h$, the product $hg$ is in $H_i$. Then we require that $R(s, v)$ holds in $F$ if and only if $R((\rho(h, s))_{s \in S}, (v_{hg})_{g \in h})$ does.

Clearly this is exactly what is needed to prove the Lemma with $S' = S \cup \{v_h \mid h \in H\}$ and $\bar{\rho}$ the extension of the action; the difficulty is in seeing that achieving (2) is possible. We now elucidate this.

Suppose $v_{h_0}, \ldots, v_{h_{i-1}}$ have been defined satisfying conditions (1) and (2); we shall describe how to choose an appropriate $v_{h_i}$. To ground the
argument, choose an arbitrary \( w \in V \setminus (S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\}) \). Consider the substructure \( B \) of \( F \) on \( |B| = S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\} \cup \{w\} \). It lies in \( K \), and \( \bar{\rho} \) (with \( w \) as \( v_{h_i} \)) acts as a partial action on its underlying set, but probably does not respect its relations. If we can demonstrate how to change \( B \) to another structure \( \bar{B} \) in \( K \) for which \( \bar{\rho} \) respects the relations, without changing the substructure on \( S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\} \), then weak homogeneity will allow us to choose an appropriate \( v_{h_i} \in V \setminus (S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\}) \) so that \( F \upharpoonright S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\} \) is isomorphic to \( \bar{B} \), whence (2) will be satisfied. With this in mind we temporarily shift our perspective, thinking of our underlying set \( |B| = S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\}, w = v_{h_i} \) as fixed but the relations satisfied by \( B \) as being mutable, so long as we remain in \( K \) and do not change the induced substructure on \( S \cup \{v_{h_0}, \ldots, v_{h_{j-1}}\} \).

We change our relations \( R \) by induction on the arity of \( R \); thanks to General Irreflexivity, this process will terminate after \( |B| \) stages, giving a \( \bar{B} \) for which \( \bar{\rho} \) respects the relations. So suppose we have a \( B_m \in K \) such that for all \( l < m \) and \( R \in L \) of arity \( l \), \( \bar{\rho} \) respects \( R \), that is, for all \( h \in H \), \( R(t_1, \ldots, t_l) \) if and only if \( R(\bar{\rho}(h, t_1), \ldots, \bar{\rho}(h, t_l)) \) whenever all of the \( v_g \) terms involved lie in \( \{v_{h_0}, \ldots, v_{h_i}\} \).

Enumerate the \( m \)-element subsets of \( |B| \) containing \( v_{h_i} \) as \( X_1, \ldots, X_{|B| m-1} \). Similarly to the proof of Lemma 18, we proceed to choose the \( m \)-ary relations that hold for \( X_j \) by induction on \( j \), building structures \( B^j \) with the same \( (m-1) \)-frame as \( B_{m-1} \), and the relations on \( X_k \) for \( 1 \leq k \leq j \) which we shall specify in such a way that \( \bar{\rho} \) respects \( m \)-ary relations on these \( X_k \) and other \( m \)-element subsets of \( |B| \setminus \{v_{h_i}\} \).

For the base case take \( B^0_m = B_m \). Now suppose we have constructed \( B^{j-1}_m \). For each \( h \in H \), consider \( hX_j = \{\bar{\rho}(h, x) \mid x \in X_j\} \). If there is an \( h \) such that \( hX_j \subset (|B| \setminus \{v_{h_i}\}) \) or \( hX_j = X_{j'} \) for some \( j' < j \), we take \( B^j_m \) to be a structure \( B' \) as given by the Adoptive Property for \( B = B^{j-1}_m, A = B^{j-1}_m \setminus X_j \) and \( A' = B^{j-1}_m \setminus hX_j \). That is, the \( m \)-ary relations on \( X_j \) in \( B^j_m \) are given by

\[
R(x_1, \ldots, x_m) \leftrightarrow R(\bar{\rho}(h, x_1), \ldots, \bar{\rho}(h, x_m))
\]

whenever \( \{x_1, \ldots, x_m\} = X_j \): this is precisely what is required to make \( \bar{\rho}(h, \cdot) \) respect \( R \). The requirement on frames for the Adoptive Property holds by the inductive assumption on \( m \) and the fact that \( (B^{j-1}(m-1)) = B^{(m-1)}_m \).

We also have that \( B^j_m \) is well-defined, because if both \( hX_j \) and \( h'X_j \) are in \( |B| \setminus \{v_{h_i}\} \) m \cup \{X_1, \ldots, X_{j-1}\} \), then \( \bar{\rho}(h^{-1}hX_j) = h'X_j \), and so by the inductive assumption on \( j \), the \( m \)-ary relations on \( hX_j \) and \( h'X_j \) agree.
Finally, if there is no such $h$, our choice of the $m$-ary relations on $X_j$ does not matter, and we may take $B^j_m = B^j_{m-1}$. This completes the recursive construction of $B^j_m$ over $j$, and hence of $B_m$ over $m$, and hence of $\bar{B} \in \mathcal{K}$ with $|\bar{B}| = |B|$ and $\bar{B}[|B| \setminus \{v_{h_i}\}] = B[|B| \setminus \{v_{h_i}\}]$ such that $\tilde{\rho}$ respects the relations in $\bar{B}$. As described above, the weak homogeneity of $F$ now lets us choose a $v_{h_i} \in V$ such that $F|S \cup \{v_{h_0}, \ldots, v_{h_i}\}$ is isomorphic to $\bar{B}$, and we are done.

\[ \text{Corollary 35. Let } G \text{ be a metrizable profinite group; then } G \text{ has a continuous faithful action on } F. \]

\[ \text{Proof. Recall that a group is a metrizable profinite group if and only if there is a sequence } (H_n)_{n \geq 1} \text{ of normal open subgroups of } G \text{ of finite index such that } H_1 \supset H_2 \supset \cdots \supset H_n \supset \cdots \text{ and } G = \lim_{\leftarrow} G/H_n. \text{ By induction on } n \text{ we construct a sequence of sets } S_n \subset V \text{ of vertices of } F \text{ such that} \]

- $S_n \subset S_{n+1}$;
- there is a faithful action $\rho_n$ of $G/H_n$ on $F_{S_n}$;
- $\cup_n S_n = V$.

Let $V = \{v_i \mid i \in \mathbb{N}\}$ be an enumeration of $V$. For $n = 0$ let $H_0 = G$ and $S_0 = \{v_0\}$ with the trivial action of the trivial group $G/H_0$. Suppose that $n \geq 1$ and that we already defined $S_{n-1}$ and a faithful action $\rho_{n-1}$ of $G/H_{n-1}$ on $S_{n-1}$. Apply Lemma 34 with $H = G/H_n$ and $S = S_{n-1}$, the action $\rho$ factoring through $\rho_{n-1}$ and $v$ being the vertex of $V \setminus S_{n-1}$ with least index. We thus obtain a faithful action $\rho_n$ of $G/H_n$ on $S' := S_n$.

Define an action $\bar{\rho}$ of $G$ on $F$ by $\bar{\rho}(\bar{g}, v_n) = \rho_n(g, v_n)$, where $g$ is the image of $\bar{g}$ in $G/H_n$. The action $\bar{\rho}$ is faithful, since an element acting trivially on every $v \in V$ is contained in $H_n$ for every $n \geq 1$. Because $G = \lim_{\leftarrow} (G/H_n)$, we have $\bigcap H_n = (e)$, and are done. \qed

\[ \text{Lemma 36. Let } G \text{ be a finite group, } H \subset G \text{ a subgroup and } S \subset V \text{ a finite subset. Suppose that } \rho: H \times (F \upharpoonright S) \to F \upharpoonright S \text{ is a left action with trivial stabilizers. Then there is a finite subset } S' \subset V \text{ with } S' \supset S \text{ and an action } \rho': G \times F \upharpoonright S' \to F \upharpoonright S' \text{ with trivial stabilizers such that } \rho'|_{H \times F \upharpoonright S} = \rho. \]

\[ \text{Proof. Let } n \text{ be the number of orbits of } H \text{ on } S. \text{ Identify } S \text{ with its action of } H \text{ with the subset } H \times \{1, \ldots, n\} \subset G \times \{1, \ldots, n\} \text{ with the trivial action of} \]

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on the second component. Exactly as in the proof of Lemma 34, we may construct a structure \( \bar{B} \) in \( K \) on underlying set \( G \times \{1, \ldots, n\} \) such that the natural \( G \)-action (which extends the \( H \)-action on \( H \times \{1, \ldots, n\} \)) respects the relations of \( \bar{B} \). Indeed the argument proceeds by induction over \( m \) on the \( m \)-frame, and for each \( m \) through an induction over \( j \) on an enumeration \( (X_j) \) of \( [G \times \{1, \ldots, n\}]^m \setminus [H \times \{1, \ldots, n\}]^m \). By weak homogeneity, the embedding of \( H \times \{1, \ldots, n\} \sim S \) into \( F \) extends to an embedding of \( \bar{B} \) into \( F \), and we may take \( S' \) to be the range of this embedding and \( \rho' \) to be the induced \( G \)-action.

\[ \begin{align*}
\text{Corollary 37.} & \quad \text{Let } G \text{ be the direct limit of a sequence of injections of finite groups } G_n \hookrightarrow G_{n+1} \text{ for } n \geq 1. \text{ Then } G \text{ has an action on } F \text{ with trivial stabilizers.} \\
\text{Proof.} & \quad \text{Let } G_0 \text{ be the trivial group. We prove by induction that for all } n \geq 0 \\
& \quad \text{there is a subset } S_n \subset V \text{ with a left action } \rho_n : G_n \times F | S_n \to F | S_n \text{ with trivial stabilizers such that for all } n \geq 1 \text{ we have} \\
& \quad \bullet \ S_{n-1} \subset S_n, \\
& \quad \bullet \ \rho_n |_{G_{n-1} \times F | S_{n-1}} = \rho_{n-1} \text{ and} \\
& \quad \bullet \ \bigcup S_n = V. \\
\end{align*} \]

Let \( V = \{v_i \mid i \in \mathbb{N}\} \) be an enumeration of \( V \). For \( n = 0 \) let \( S_0 = \{v_0\} \) and \( \rho_0 : G_0 \times F_{S_0} \to F_{S_0} \) be the unique action. Suppose that \( n \geq 1 \) and that \( S_{n-1} \) and \( \rho_{n-1} \) have been defined; let \( v \in V \setminus S_{n-1} \) be the element with least index. Apply Lemma 34 to obtain a finite \( S' \supset S \cup \{v\} \) and an action \( \rho' \) of \( G_{n-1} \) on \( F_{S'} \) with trivial stabilizers. Apply Lemma 36 to \( \rho' \) to obtain a subset \( S_n \subset V \) and an action \( \rho_n : G_n \times F_{S_n} \to F_{S_n} \) with the required properties.

This means that for \( n > 0 \), subgroups of \( GL_n \) over the algebraic closure of a finite field are subgroups of \( \text{Aut}(F) \). So too are \( S_\infty \) and \( \mathbb{Q}/\mathbb{Z} \). Indeed, in each of these cases, it is clear that every finite subset is contained in a finite subgroup.

Corollary 37 allows us to embed many divisible groups into \( \text{Aut}(F) \). Note however that \( \text{Aut}(F) \) itself is not divisible. To see this, consider the following construction of an element of \( G \) of order 4 not admitting a square root. Take distinct vertices \( v \) and \( w \) in \( V \) and let a generator \( u \in \mathbb{Z}/4\mathbb{Z} \) act on \( \{v, w\} \) non-trivially. Use Lemma 34 inductively to extend this to a \( \mathbb{Z}/4\mathbb{Z} \) action.

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on \( F \) with trivial stabilizers on \( V \setminus \{v, w\} \). Suppose for contradiction that \( g \in \text{Aut}(F) \) is such that \( g^2 = u \in \text{Aut}(F) \). Then \( u(g(v)) = g(u(v)) = g(w) \) and \( u(g(w)) = g(v) \), and therefore \( g(v) \) is an element of \( V \) with non-trivial stabilizer in \( \mathbb{Z}/4\mathbb{Z} \), and thus \( g \) permutes \( \{v, w\} \). Hence \( g^2(v) = v \), contradicting \( g^2(v) = u(v) = w \).

7 Topology

In this section we establish the homeomorphism type of the random simplicial complex: despite its random construction, the geometric realisation of the random simplicial complex is topologically quite simple.

In this section we shall always give a name for the underlying set of a simplicial complex, freeing up the notation \(|\cdot|\) to represent the geometric realisation, defined as follows. Let \( S \) be a simplicial complex on a set \( V \). Let \( e_v \) for \( v \in V \) be the standard basis of \( \mathbb{R}^V \). The geometric realisation \(|S|\) of \( S \) is the union over faces \( F \in S \) of the convex hull of the set \( \{e_v \mid v \in F\}\). A subset \( A \subset |S| \) is taken to be open whenever \( A \cap \mathbb{R}^W \) is open for all finite \( W \subset V \). We sometimes identify a vertex \( v \) of \( S \) and the corresponding element \( e_v \) in the realisation of \( S \).

**Definition 38.** Let \( n \) be a natural number. The \( n \)-simplex \( \Delta_n \) on \( \{0, 1, \ldots, n\} \) is the simplicial complex consisting of all subsets of \( \{0, 1, \ldots, n\} \); the standard \( n \)-simplex is the geometric realisation of \( \Delta_n \). The infinite simplex \( \Delta_\infty \) on \( \mathbb{N} \) is the simplicial complex consisting of all finite subsets of \( \mathbb{N} \), that is \( \Delta_\infty := [\mathbb{N}]^{fin} \).

Let \( n \) be an element of \( \mathbb{N} \cup \{\infty\} \). A **piecewise linear map** of a simplicial complex \( S \) to \( \mathbb{R}^n \) is a function from the geometric realisation of \( S \) to \( \mathbb{R}^n \) that is linear on every face of \( S \). Thus a piecewise linear function is uniquely determined by its values on the vertices of the geometric realisation of \( S \).

A simplicial complex \( S \) is a **cone with vertex** \( a \) if whenever \( F \) is a face of \( S \) also \( F \cup \{a\} \) is a face of \( S \).

**Lemma 39.** Let \( n \) be a natural number, let \( V \) be a set and let \( v \in V \). Suppose that \( S \) is a simplicial complex on \( V \setminus \{v\} \) and that \( S' \) is a simplicial complex on \( V \) containing \( S \) as an induced subcomplex. If \( \varphi: |S| \to \mathbb{R}^n \) is a piecewise linear map to \( \mathbb{R}^n \), then there is a unique piecewise linear map \( \varphi': |S'| \to \mathbb{R}^{n+1} \) such that \( \varphi'(e_v) = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \) and for all \( y \in |S| \)
we have $\varphi'(y) = (\varphi(y), 0) \in \mathbb{R}^{n+1}$. If the map $\varphi$ is a homeomorphism onto its image, then the same is true for the extension $\varphi'$.

**Proof.** Every point $x$ in $|S'|$ is in the convex hull of $\{e_w \mid w \in F\}$ for some face $F$ of $S'$, and hence may be written as $te_v + (1 - t)y$ for some $t \in [0, 1]$ and $y \in |S|$. It is then straightforward to check that defining $\varphi'(x) = ((1 - t)\varphi(y), t)$ yields a well-defined, piecewise linear map $\varphi': |S'| \to F^{n+1}$, that is a homeomorphism to its image if $\varphi$ is.

**Lemma 40.** Let $V$ be a finite set, let $S$ be a simplicial complex on $V$ and let $v$ be a vertex of $S$. Suppose that $n$ is a natural number and $\varphi: |S| \to \mathbb{R}^{n+1}$ is a piecewise linear map such that $\varphi(e_v) = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ and $\varphi|_{|S_V \setminus \{v\}|}$ is a homeomorphism to the standard $(n - 1)$-simplex. There exists a finite set $V'$ containing $V$, a simplicial complex $S'$ on $V'$ and a piecewise linear map $\varphi': |S'| \to \mathbb{R}^{n+1}$ such that

- the complex $S$ is the induced subcomplex of $S'$ on $V$,
- the piecewise linear map $\varphi'$ is a homeomorphism to the standard $n$-simplex in $\mathbb{R}^{n+1}$, and

  - the restriction of $\varphi'$ to $|S|$ coincides with $\varphi$.

**Proof.** Let $(F_1, F_2, \ldots, F_r)$ be a list of those faces $F$ of $S$ such that $F \cup \{v\}$ is not a face of $S$, ordered by increasing dimension. Note in particular that for every proper subset $X \subsetneq F_1$ we have that $X \cup \{v\}$ is a face of $S$. Let $S_1$ be the simplicial complex obtained by adding to $S$ a new vertex $v_1$ and having $\{v_1\} \cup G$ as a face if and only if $G \subseteq F_1 \cup \{v\}$. (Thus $S_1$ is obtained by adding to $S$ the cone with apex $v_1$ over the boundary of the simplex $F_1 \cup \{v\}$, and the simplex $F_1 \cup \{v\}$ is not a face of $S$.) We extend $\varphi$ to a piecewise linear map $\varphi_1$ of $|S_1|$ into $|\Delta_n|$ by mapping $v_1$ to the barycentre $b_1$ of $\varphi(|F_1 \cup \{v\}|)$. Observe that the image of the extension $\varphi_1$ contains the convex hull of $\varphi(F_1 \cup \{v\})$.

Now suppose that $i \geq 2$ and that $S_{i-1}$ and $\varphi_{i-1}$ are already defined. Let $v_{i_1}, \ldots, v_{i_s}$ be the vertices of $S_{i-1} \setminus S$ that lie on the cone in $\mathbb{R}^{n+1}$ with vertex $\varphi(e_v)$ and base the image under $\varphi$ of the boundary of $|F_i|$; denote by $b_{i_1}, \ldots, b_{i_s}$ the images under $\varphi_{i-1}$ of $v_{i_1}, \ldots, v_{i_s}$ respectively. Let $S_i$ be the simplicial complex obtained by adding a new vertex $v_i$ to $S_{i-1}$ and adding to $S_{i-1}$ the cone with vertex $v_i$ on the simplicial complex induced by $S_{i-1}$ on $F_i \cup \{v\} \cup \{v_{i_1}, \ldots, v_{i_s}\}$. We extend $\varphi_{i-1}$ to a piecewise linear map $\varphi_i: |S_i| \to \mathbb{R}^{n+1}$ by setting $\varphi_i(b_{i_1}, \ldots, b_{i_s}) = (1, \ldots, 1, 0, 1)$ for each $1 \leq s \leq i$ and $\varphi_i|_{|S_{i-1} \setminus S|}$ $\varphi_{i-1}$, and $\varphi_i|_{|S_i \setminus S_{i-1}|}$ as a linear combination of $\varphi_i(b_{i_1}, \ldots, b_{i_s})$ and $\varphi_{i-1}$.

Now observe that $\varphi_i|_{|S_i \setminus S_{i-1}|}$ is a homeomorphism onto the boundary of $|\Delta_n|$ if $\varphi_{i-1}$ is a homeomorphism to its image. Since $\varphi_i|_{|S_i \setminus S_{i-1}|}$ is a homeomorphism to its image, we have that $\varphi_i|_{|S_i \setminus S_{i-1}|}$ is a homeomorphism onto its image, and hence $\varphi_i$ is a homeomorphism onto its image.
by letting $\varphi_i(v_i)$ be the barycentre $b_i$ of the set consisting of $v$ and the vertices of $\varphi_{i-1}(|F_i|)$. Observe that the image of the extension $\varphi_i$ contains the convex hulls for $j \leq i$ of $\varphi(F_j \cup \{v\})$, and that $\varphi_i$ is a homeomorphism onto its image. Continuing in this manner, we conclude that the simplicial complex $S' := S_r$ and the piecewise linear map $\varphi' := \varphi_r$ satisfy the requirements and the proof is complete.

Theorem 41. The geometric realisation of the random simplicial complex $F_{\text{SC}}$ is homeomorphic to the realisation of the infinite simplex $\Delta_\infty$.

Proof. For each $n \in \mathbb{N}$ we construct by induction on $n$ a finite induced simplicial complex $S_n \subset F_{\text{SC}}$ and a piecewise linear map $\varphi_n : |S_n| \to \mathbb{R}^{n+1}$ such that

- $S_n \subset S_{n+1}$ and $\cup_n S_n = F_{\text{SC}}$;
- $\varphi_n$ is a piecewise linear homeomorphism to the standard $n$-simplex in $\mathbb{R}^{n+1}$;
- the restriction of $\varphi_{n+1}$ to $|S_n|$ coincides with $\varphi_n$ followed by the inclusion of $\mathbb{R}^{n+1}$ into $\mathbb{R}^{n+2}$ as the linear subspace with vanishing last coordinate.

Fix a bijection between the vertex set of $F_{\text{SC}}$ and $\mathbb{N}$, thus identifying the vertex set of $F_{\text{SC}}$ with $\mathbb{N}$. For $n = 0$ we let $S_0$ be the vertex 0 of $F_{\text{SC}}$; thus $|S_0|$ is the standard 0-simplex and we let $\varphi_0$ be the inclusion of $|S_0|$ in its geometric realisation in $\mathbb{R}^1$.

Suppose that $S_{n-1}$ and $\varphi_{n-1}$ have already been defined and let $v \in \mathbb{N}$ be the least vertex not already appearing in $S_{n-1}$. Let $S \subset F_{\text{SC}}$ be the simplicial complex induced on the vertices of $S_{n-1}$ and $v$. Using Lemma 39, extend $\varphi_{n-1}$ to a piecewise linear map $\varphi : |S| \to \mathbb{R}^{n+1}$ by defining $\varphi(v) = (0, \ldots, 0, 1)$, and for each vertex $w$ of $S_{n-1}$, defining $\varphi(w) = (\varphi_{n-1}(w), 0)$.

We are therefore in a position to apply Lemma 40 to the simplicial complex $S$ and the piecewise linear map $\varphi$: let $S'$ and $\varphi'$ be as in that lemma. By weak homogeneity (see Definition 7) of $F_{\text{SC}}$, there is an induced subcomplex $S_n$ of $F_{\text{SC}}$ containing $S_{n-1}$ and an isomorphism $\iota : S_n \to S'$ such that $\iota$ is the identity on $S_{n-1}$. Let $\varphi_n : |S_n| \to \mathbb{R}^{n+1}$ be the the piecewise linear map defined by $\varphi_n(w) = \varphi'(\iota(w))$ for each vertex $w \in S_n$. The complex $S_n$ and the map $\varphi_n$ satisfy the requirements, and the theorem follows. \qed
We do not expect contractibility to be a first order property in our language.

**Question 42.** Is it the case that almost all finite simplicial complexes have contractible realisation (under the frame-wise uniform measure)?

It follows from Theorems 32 and 41 that a negative answer to Question 42 would imply that contractibility is not a first order property.

## 8 Acknowledgement

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## References


