Colimit preservation from weaker large cardinals

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Background

**Theorem (Rosický, Trnková & Adámek, 1990)**

Assuming Vopěnka’s Principle, for each full embedding $F : \mathcal{A} \to \mathcal{K}$ with $\mathcal{K}$ an accessible category, there is a regular cardinal $\lambda$ such that $F$ preserves $\lambda$-directed colimits.

Recall that a poset is $\lambda$-directed if every subset of cardinality less than $\lambda$ has an upper bound. A $\lambda$-directed diagram is one whose index category is a $\lambda$-directed poset.
Vopěnka’s Principle

This is a very strong set-theoretic axiom schema.

**Vopěnka’s Principle (VP)**

For any signature $\Sigma$, and any proper class $C$ of $\Sigma$-structures, there are distinct structures $A$ and $B$ in $C$ such that there exists a homomorphism from $A$ to $B$. 
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Bagaria, Casacuberta, Mathias and Rosický: VP for classes defined by formulae of a given quantifier complexity is strictly weaker than full VP, so many specific applications of VP can be obtained from weaker assumptions.
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**Question:**
Can the colimit preservation theorem from the previous slide be stratified in this way?
Answer

Yes!
Answer

Yes!
Theorem (Bagaria & B-T)

Suppose that $\mathcal{K}$ is a full subcategory of $\text{Str } \Sigma$ for some signature $\Sigma$. Let $F : \mathcal{A} \rightarrow \mathcal{K}$ be any $\Sigma_n$-definable full embedding with $\Sigma_n$-definable domain category $\mathcal{A}$, for some $n > 0$. If there exists a $C^{(n)}$-extendible cardinal greater than

- the rank of $\Sigma$,
- the arity of each function or relation symbol in $\Sigma$, and
- the ranks of the parameters used in some $\Sigma_n$ definitions of $F$ and $\mathcal{A}$ and in some definition of $\mathcal{K}$,

then there exists a regular cardinal $\lambda$ such that $F$ preserves $\lambda$-directed colimits.
The set-theoretic framework

The von Neumann hierarchy

\[ V_0 = \emptyset \]
\[ V_{\alpha+1} = \mathcal{P}(V_\alpha) \]
\[ V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha \quad \text{for limit ordinals } \lambda \]
\[ V = \bigcup_{\alpha \in \text{Ord}} V_\alpha, \quad \text{the full set-theoretic universe.} \]

The rank of a set \( x \) is the least \( \alpha \) such that \( x \subseteq V_\alpha \).
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Classes are collections of sets given by formulae (possibly with parameters): \( C = \{ x \mid \varphi(x, p) \} \) for some formula \( \varphi \) and set \( p \).
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Categories and functors are taken to be classes.
Formula complexity

Levy hierarchy
In the language of set theory, \( \Sigma = \{ \in \} \), a formula is

- \( \Sigma_0 \) and \( \Pi_0 \) if all of its quantifiers are bounded (i.e., of the form \( \forall x \in X \) or \( \exists x \in X \)).
- \( \Sigma_{n+1} \) if it is of the form \( \exists x (\varphi(x)) \) for some \( \Pi_n \) formula \( \varphi \).
- \( \Pi_{n+1} \) if it is of the form \( \forall x (\varphi(x)) \) for some \( \Sigma_n \) formula \( \varphi \).

A class (or category, or functor) is \( \Sigma_n \) if there is a \( \Sigma_n \) formula defining it.
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- $\Sigma_{n+1}$ if it is of the form $\exists x(\varphi(x))$ for some $\Pi_n$ formula $\varphi$.
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A class (or category, or functor) is $\Sigma_n$ if there is a $\Sigma_n$ formula defining it.

For a structure $\mathcal{M}$, we write $\mathcal{M} \models \varphi(m)$ for “$\mathcal{M}$ satisfies formula $\varphi$ with parameter $m$”.

Example

$$\langle \mathbb{Z}, + \rangle \models \forall x \exists y(x + y = 3)$$
We denote by $C^{(n)}$ the class of cardinals $\kappa$ such that $V_\kappa \prec_{\Sigma_n} V$, that is, for every $\Sigma_n$ formula $\varphi(x)$ and set $x_0 \in V_\kappa$,

$$\langle V_\kappa, \in \rangle \models \varphi(x_0) \text{ if and only if } \langle V, \in \rangle \models \varphi(x_0).$$

For every $n$, $C^{(n)}$ is unbounded: given any cardinal $\gamma$, one can find a cardinal $\kappa$ greater than $\gamma$ in $C^{(n)}$.

Proof sketch

By induction on $n$. Iteratively take larger and larger $\kappa$ in $C^{(n-1)}$ so that $V_\kappa$ contains sets witnessing statements of the form $\exists x (\varphi(x))$ with $\varphi$ a $\Pi_{n-1}$ formula. This process “closes off” at a limit point $\kappa$ in $C^{(n)}$.

Note however that trying this for all formulae (i.e., all $n$) at once raises Gödelian, definability of definability problems.
$C^{(n)}$ cardinals

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**Note** however that trying this for all formulae (i.e., all $n$) at once raises Gödelian, definability of definability problems.
Recall that an elementary embedding is a function preserving all formulae.

**Definition**
A cardinal $\kappa$ is $C^{(n)}$-extendible if for every $\lambda > \kappa$ there is a cardinal $\mu > \lambda$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that

1. $\kappa = \text{crit}(j)$, i.e., $\kappa$ is the least ordinal such that $j(\kappa) \neq \kappa$,
2. $j(\kappa) > \lambda$, and
3. $j(\kappa) \in C^{(n)}$. 

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2. $j(\kappa) > \lambda$, and
3. $j(\kappa) \in C^{(n)}$.

$\kappa$ is $C^{(n)+}$-extendible if moreover, for every $\lambda > \kappa$ in $C^{(n)}$, there is a $\mu > \lambda$ in $C^{(n)}$ and an elementary embedding $j : V_\lambda \to V_\mu$ such that (1), (2) and (3) hold.
Theorem (Bagaria & B-T)

For all $\alpha$,

$$\exists \kappa > \alpha (\kappa \text{ is } C^{(n)}\text{-extendible})$$

is equivalent to

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Theorem (Bagaria, Casacuberta, Mathias & Rosický)

Vopěnka’s Principle is equivalent to the existence of a proper class of $C^{(n)+}$-extendible cardinals for every $n$. Moreover, the existence of a $C^{(n)+}$-extendible $\kappa$ corresponds exactly to Vopěnka’s Principle for classes that are $\Sigma_{n+2}$-definable with parameters from $V_\kappa$. 
The main theorem again

Theorem (Bagaria & B-T)

Suppose that $\mathcal{K}$ is a full subcategory of $\text{Str} \; \Sigma$ for some signature $\Sigma$. Let $F : \mathcal{A} \to \mathcal{K}$ be any $\Sigma_n$-definable full embedding with $\Sigma_n$-definable domain category $\mathcal{A}$, for some $n > 0$. If there exists a $C^{(n)}$-extendible cardinal greater than

i. the rank of $\Sigma$,

ii. the arity of each function or relation symbol in $\Sigma$, and

iii. the ranks of the parameters used in some $\Sigma_n$ definitions of $F$ and $\mathcal{A}$ and in some definition of $\mathcal{K}$,

then there exists a regular cardinal $\lambda$ such that $F$ preserves $\lambda$-directed colimits.
Want to show $F : \mathcal{A} \to \mathcal{K}$ preserves $\lambda$-directed colimits.

Sufficient:

\[ i \circ F : \mathcal{A} \to \text{Str } \Sigma \text{ preserves } \lambda\text{-directed colimits,} \]

where $i : \mathcal{K} \to \text{Str } \Sigma$ is the inclusion functor (and this notional inclusion doesn’t change the quantifier complexity). So WLOG assume $F : \mathcal{A} \to \text{Str } \Sigma$. 

Note: $\text{Str } \Sigma$ has all $\lambda$-directed colimits, for $\lambda$ greater than the arities of the symbols in $\Sigma$ (i.e. cardinals as per (ii)). Let $\beta$ be sufficiently large as per (i), (ii) and (iii).
Want to show $F : \mathcal{A} \to \text{Str} \Sigma$ preserves $\lambda$-directed colimits.

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Let $\beta$ be sufficiently large as per (i), (ii) and (iii).
Proof

Want to show $F : \mathcal{A} \to \mathbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

Consider the following category $\mathcal{C}$:

**Objects:** $\mathbf{Str} \Sigma$ morphisms $a : \bar{A} \to F(A)$ such that for some $\lambda > \beta$ and some $\lambda$-directed diagram $D$ in $\mathcal{A}$,
- $A$ is the colimit of $D$ in $\mathcal{A}$,
- $\bar{A}$ is the colimit of $F D$ in $\mathbf{Str} \Sigma$, and
- $a$ is the morphism induced by the image under $F$ of the $\mathcal{A}$-colimit cocone from $D$ to $A$.

**Morphisms:** From $a$ to $b$: pairs $\langle g, h \rangle$ of $\mathbf{Str} \Sigma$ morphisms such that

\[
\begin{array}{ccc}
\bar{A} & \xrightarrow{a} & F(A) \\
\downarrow{g} & & \downarrow{h} \\
\bar{B} & \xrightarrow{b} & F(B).
\end{array}
\]

commutes.
Proof

Want to show $F : \mathcal{A} \to \text{Str } \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

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$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$
Proof

Want to show $F : \mathcal{A} \to \mathbf{Str} \Sigma$ preserves $\lambda$-directed colimits. Let $C$ be the category of $\lambda$-directed colimit morphisms $a : \bar{\mathcal{A}} \to F(\mathcal{A})$.

Let $C^*$ be the full subcategory of $C$ of those $a$ which are not isomorphisms.

If the theorem fails, then $C^*$ is not essentially small.
Proof

Want to show $F : \mathcal{A} \to \text{Str } \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{\mathcal{A}} \to F(\mathcal{A})$

$\mathcal{C}^*$: full subcat. of non-isos

Want to show $\mathcal{C}^*$ is essentially small

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**Proof**

Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

- $\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \tilde{A} \to F(A)$
- $\mathcal{C}^*$: full subcat. of non-isos

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**Claim**

Obj($\mathcal{C}^*$) is $\Sigma_{n+2}$-definable over the language of set theory (extended with $\mathcal{P}_\beta$):
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$\text{Obj}(\mathcal{C}^*)$ is $\Sigma_{n+2}$-definable over the language of set theory (extended with $P_\beta$): $a \in \text{Obj}(\mathcal{C}^*)$ iff

$$\exists \lambda \exists \mathcal{D} \exists \langle \tilde{A}, \tilde{\eta} \rangle \exists \langle A, \eta \rangle (\lambda \text{ is a regular cardinal } \land \mathcal{D} \text{ is a diagram in } \mathcal{A} \land \mathcal{D} \text{ is } \lambda\text{-directed} \land \langle \tilde{A}, \tilde{\eta} \rangle = \text{Colim}_{\text{Str } \Sigma}(F\mathcal{D}) \land \langle A, \eta \rangle = \text{Colim}_{\mathcal{A}}(\mathcal{D}) \land a : \tilde{A} \to F(A) \text{ is the induced homomorphism} \land a \text{ is not an isomorphism}).$$

The universal property of colimits makes the middle line $\Pi_{n+1}$. 
Proof

Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

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$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

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Assume for contradiction that $\mathcal{C}^*$ is not essentially small.
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Want to show $\mathcal{C}^*$ is essentially small

Assume for contradiction that $\mathcal{C}^*$ is not essentially small.

Let $\kappa$ be a $C^{(n)+}$-extendible cardinal greater than $\beta$.

Let $a$ be an object of $\mathcal{C}^*$ of rank $> \kappa$, arising from a $\lambda_a$-directed diagram $D_a$ for some $\lambda_a > \kappa$.

Let $\lambda \in C^{(n)}$ be greater than the ranks of $a, D_a, F D_a$, and the corresponding cocones $\langle \tilde{A}, \tilde{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$. 
Proof

Want to show $F : \mathcal{A} \to \text{Str} \Sigma$ preserves $\lambda$-directed colimits.

$C$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

$C^*$: full subcat. of non-isos; $C^*$ is $\Sigma_{n+2}$-definable

Want to show $C^*$ is essentially small

If not take $\kappa$ a $C^{(n)^+}$-extendible, $a \in C^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger

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Want to show $F : \mathcal{A} \to \text{Str } \Sigma$ preserves $\lambda$-directed colimits.

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Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger

Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $C^{(n)}$.

Then $V_\mu \models \lambda a$, $D a$, $\langle \bar{A}, \bar{\eta} \rangle a$ and $\langle A, \eta \rangle a$ witness that $a \in \text{Obj}(\mathcal{C}^*)$.

Henceforth work in $V_\mu$.

Note that because $\kappa > \beta$, the definition of $F$ is unaffected by $j$, so $j$ commutes with $F$. 
Proof

Want to show $F : A \to \text{Str } \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \tilde{A} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger

Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $C^{(n)}$. 
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Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

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Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $C^{(n)^+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger,

$j : V_\lambda \to V_\mu$ a $C^{(n)^+}$-extendibility embedding

Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $C^{(n)}$. 
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Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

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Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $\mathcal{C}(n)^+\text{-extendible}$, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in \mathcal{C}(n)$ yet bigger,

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Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $\mathcal{C}(n)$. Then

$$V_\mu \models \lambda_a, \mathcal{D}_a, \langle \tilde{A}, \tilde{\eta} \rangle_a \text{ and } \langle A, \eta \rangle_a \text{ witness that } a \in \text{Obj}(\mathcal{C}^*).$$
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Want to show $F : \mathcal{A} \to \text{Str} \Sigma$ preserves $\lambda$-directed colimits.

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If not take $\kappa$ a $\mathcal{C}^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in \mathcal{C}^{(n)}$ yet bigger,

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Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $\mathcal{C}^{(n)}$. Then

$V_\mu \models \lambda_a, \mathcal{D}_a, \langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$ witness that $a \in \text{Obj}(\mathcal{C}^*)$.

Henceforth work in $V_\mu$. 
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$j : V_\lambda \to V_\mu$ a $C^{(n)+}$-extendibility embedding

Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$ such that $\mu > j(\kappa) > \lambda$ are all in $C^{(n)}$. Then

$V_\mu \models \lambda_a, D_a, \langle \bar{A}, \bar{\eta} \rangle_a$ and $\langle A, \eta \rangle_a$ witness that $a \in \text{Obj}(\mathcal{C}^*)$.

Henceforth work in $V_\mu$.

Note that because $\kappa > \beta$, the definition of $F$ is unaffected by $j$, so $j$ commutes with $F$. 
Proof

Want to show $F : A \to \text{Str } \Sigma$ preserves $\lambda$-directed colimits.

$C$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

$C^*$: full subcat. of non-isos; $C^*$ is $\Sigma_{n+2}$-definable

Want to show $C^*$ is essentially small

If not take $\kappa a C^{(n)+}$-extendible, $a \in C^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger,

$j : V_\lambda \to V_\mu$ a $C^{(n)+}$-extendibility embedding
Proof

Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \tilde{A} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $\mathcal{C}^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in \mathcal{C}^{(n)}$ yet bigger,

$j : V_\lambda \to V_\mu$ a $\mathcal{C}^{(n)+}$-extendibility embedding

Since $j$ is elementary, we have a morphism in $\mathcal{C}^* V_\mu$

\[
\begin{array}{ccc}
\tilde{A} & \xrightarrow{a} & F(A) \\
\downarrow j|\tilde{A} & & \downarrow j|F(A) \\
\downarrow & & \\
j(\tilde{A}) & \xrightarrow{j(a)} & j(F(A)).
\end{array}
\]
Proof

Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{\mathcal{A}} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $C^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in C^{(n)}$ yet bigger,

$j : V_\lambda \to V_\mu$ a $C^{(n)+}$-extendibility embedding

Since $j$ is elementary, we have a morphism in $\mathcal{C}^* V_\mu$

$$
\begin{array}{c}
\bar{\mathcal{A}} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
j(\bar{\mathcal{A}}) \\
\end{array} \\
\xrightarrow{a} \\
\xrightarrow{\lambda(a)} \\
\xrightarrow{\lambda(F(A))} \\
F(A) \\
\xrightarrow{j(F(A))}
$$

Now, $D_a$ is $\lambda_a$-directed, so $j(D_a)$ is $j(\lambda_a)$-directed, and $j(FD_a) = Fj(D_a)$ is $j(\lambda_a)$-directed.
Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.  

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

Want to show $\mathcal{C}^*$ is essentially small
If not take $\kappa$ a $\mathcal{C}^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in \mathcal{C}^{(n)}$ yet bigger, $j : V_{\lambda} \to V_\mu$ a $C^{(n)+}$-extendibility embedding

Since $j$ is elementary, we have a morphism in $\mathcal{C}^* \mathcal{V}_\mu$

$\bar{A} \xrightarrow{a} F(A)$

$\xymatrix{ \bar{A} \ar[r]^a \ar[d]_{j|\bar{A}} & F(A) \ar[d]^{j|F(A)} \\
 j(\bar{A}) \ar[r]_{j(a)} & j(F(A)).}$

Now, $\mathcal{D}_a$ is $\lambda_a$-directed, so $j(\mathcal{D}_a)$ is $j(\lambda_a)$-directed, and $j(F\mathcal{D}_a) = Fj(\mathcal{D}_a)$ is $j(\lambda_a)$-directed. Since $j(\lambda_a) > j(\kappa) > \lambda > |\mathcal{D}_a|$, $j"F\mathcal{D}_a$ has an upper bound $F(d_0)$ in $Fj(\mathcal{D}_a)$. 
Proof

Want to show $F : \mathcal{A} \to \textbf{Str} \Sigma$ preserves $\lambda$-directed colimits.

$\mathcal{C}$: cat. of $\lambda$-directed colimit morphisms $a : \bar{A} \to F(A)$

$\mathcal{C}^*$: full subcat. of non-isos; $\mathcal{C}^*$ is $\Sigma_{n+2}$-definable

Want to show $\mathcal{C}^*$ is essentially small

If not take $\kappa$ a $\mathcal{C}^{(n)+}$-extendible, $a \in \mathcal{C}^*$ “bigger”, $\lambda \in \mathcal{C}^{(n)}$ yet bigger, $j : V_\lambda \to V_\mu$ a $\mathcal{C}^{(n)+}$-extendibility embedding