Linear Algebra Sheet 9

May 10, 2005

1. Let A be the matrix
$$\begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 3 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$
and let $B\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be the symmetric bilinear form that it defines by $B(v, w) = vAw^T$. Find a basis of $\mathbb{R}^4$ such that the matrix of B with respect to this basis is diagonal and determine whether or not the form is positive definite.

   Let $v_1 = e_1, v_2 = e_2, v_3 = e_4$; then $B(v_i, v_j) = 0$ for $i \neq j$ and we must find a vector orthogonal with respect to B to all of these vectors. To find this we must solve
$$\begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$
and we see that $v_4 = (-3 \ -4 \ 2 \ 1)$ will do. Now the matrix of B with respect to $\{v_1, v_2, v_3, v_4\}$ is diagonal and the first three diagonal entries are $2, 1, 2$ and $B(v_4, v_4) = v_4(0 \ 0 \ 12 \ 0)^T = 24$ so the 4th diagonal entry is 24. Since all diagonal entries are +ve, the form is +ve definite.

2. Let A be the matrix
$$A = \begin{pmatrix} 2 & r & r \\ r & 2 & r \\ r & r & 2 \end{pmatrix}$$
where $r \in \mathbb{R}$ and let $B\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be the symmetric bilinear form that it defines by $B(v, w) = vAw^T$. Find a basis of $\mathbb{R}^3$ such that the matrix of B with respect to this basis is diagonal and determine for which values of r the form is positive definite.

   We see that $B(e_1, e_2 - e_3) = 0$. So let $v_1 = e_1, v_2 = e_2 - e_3$ and choose v_3 to be orthogonal with respect to B to v_1 and v_2 so that we solve
$$\begin{pmatrix} 2 & r & r \\ 0 & 2-r & r-2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

from which we see that we may take $v_3 = (-r\ 1\ 1)$*. With respect to the basis* $\{v_1, v_2, v_3\}$*, the matrix of* B *is diagonal and* $B(v_1, v_1) = 2$*,* $B(v_2, v_2) = v_2 A v_2^T = (0\ 2-r\ r-2)(0\ 1\ -1)^T = 4-2r$ *whilst* $B(v_3, v_3) = (-r\ 1\ 1)(0\ -r^2+r+2\ -r^2+r+2)^T = -2(r^2-r-2)$*. Now* B *is +ve definite iff these 3 numbers are +ve; so iff* $4-2r > 0 > r^2-r-2$ *from which we see that* B *is +ve definite iff* $-1 < r < 2$*.*