

The strength of infinitary ramseyan principles can be accessed by their densities

Andrey Bovykin, Andreas Weiermann

Abstract

We conduct a model-theoretic investigation of three infinitary ramseyan statements: Ramsey Theorem for pairs and two colours (RT_2^2), Canonical Ramsey Theorem for pairs (CRT^2) and Regressive Ramsey Theorem for pairs (RegRT^2). We prove theorems that approximate the logical strength of these principles by the strength of their finite iterations known as density principles. We then investigate their logical strength using strong initial segments of models of Peano Arithmetic, in the spirit of Paris-Kirby results. The article is concluded by a discussion of two further out-reaches of densities. One concerns further investigations of the strength of Ramsey Theorem for pairs. The other one is about the asymptotic of the standard Ramsey function R_2^2 .

Key words: unprovability, Ramsey Theorem for pairs, reverse mathematics, indicator method, strong cut.

This article stands on the crossroads of two subjects: Ramsey Theory and models of Peano arithmetic.

The story of the study of the strength of infinitary principles started with the article [6] of L.Kirby and J.Paris, where they, among other things, proved that for every $k \geq 3$, the Infinite Ramsey Theorem for dimension k has exactly the same first-order arithmetical consequences (over a weak base theory) as full Peano Arithmetic.

The method of the proof was model-theoretic, using semi-regular, regular and strong initial segments of models of arithmetic (definitions are coming later). In the class of semi-regular initial segments of a model of Peano Arithmetic, an initial segment I satisfies $I \rightarrow (I)_2^3$ if and only if I is strong, thus satisfying full Peano Arithmetic. The method of proof is closely related to the subsequent method of indicators and the discovery of PA-unprovable combinatorial statements.

¹ Both authors' research has been supported by the John Templeton Foundation

When dimension k is not fixed, the logical strength is higher: $\text{ACA}_0 + \text{IRT}$ can be axiomatized through the theory ACA'_0 (that is the theory $\text{ACA}_0 + \{\forall X \forall n X^{(n)}, \text{ the } n\text{-th Turing jump of } X \text{ exists}\}$, see [10]). This result can be found in [7]. In conjunction with other results it follows that the proof-theoretic ordinal of Infinite Ramsey Theorem is ε_ω .

The crucial notion used in the proof of this result is that of density, which allowed to formulate an iterated Paris-Harrington Principle $\text{PH}^{(n)}$ and then prove that the Π_2 -consequences of the Infinite Ramsey Theorem are the same as of the first-order theory $\text{PA} + \bigcup_{n \in \omega} \text{PH}^{(n)}$.

Throughout this article, we shall use another, weaker notion of density, originally due to J.Paris [9].

Definition 1.

We say that a set X is $0\text{-dense}(c, k)$ if $|X| > \min X + 1$. For every $n \in \omega$, X is $(n + 1)\text{-dense}(c, k)$ if for every $f: [X]^k \rightarrow c$, there is an f -homogeneous $n\text{-dense}(c, k)$ subset of X . The statement “for all a, c and k , there exists a b such that $[a, b]$ is an $n\text{-dense}(c, k)$ set”.

We study approximations of infinitary principles by their densities at other levels of logical strength. For Ramsey Theorem for pairs and two colours (RT_2^2), Canonical Ramsey Theorem for pairs (CRT^2) and Regressive Ramsey Theorem for pairs (RegRT^2) we introduce respective notions of density and show that the corresponding infinitary principle is Π_2 -conservative over the corresponding theory of the form $I\Sigma_1 + \bigcup_{n \in \mathbb{N}} \text{“for all } a \text{ there is a } b \text{ such that } [a, b] \text{ is an } n\text{-dense set”}$.

These results are motivated in particular by the long-standing open problem on determining the strength of RT_2^2 . In principle the characterization via densities allows for a purely combinatorial approach to determine the set of all provably recursive functions of RT_2^2 which is one of the major open problems in logic (see problem 13.2 in [2]).

If it was possible to show that for each n there exists a primitive function $p_n: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $a \in \mathbb{N}$ the interval $[a, p_n(a)]$ is $n\text{-dense}(2,2)$ then $\text{WKL}_0 + \text{RT}_2^2$ would not prove the totality of the Ackermann function. If it was possible to find an $n \in \mathbb{N}$ such that for all but finitely many $a \in \mathbb{N}$ the interval $[a, A(a, a)]$ is not $n\text{-dense}(2,2)$ then $\text{WKL}_0 + \text{RT}_2^2$ would prove the totality of the Ackermann function.

We note in passing that the following principle MIS which is an immediate corollary of RT_2^2 is already strong enough to generate all primitive recursive functions over WKL_0^* (see, for example, [10] for a definition). MIS stands for: “every function $F: \mathbb{N} \rightarrow \mathbb{N}$ has a weakly monotone increasing subsequence”.

RT_2^2 similarly yields the principle CAC and an infinitary tournament principle

EM whose finitary version has been investigated by Erdős and Moser. We show that CAC together with EM yields a reversal to RT_2^2 and consequently there exists another line of attack at RT_2^2 in terms of density principles. Surprisingly the density principle for EM alone is pretty weak. A closer inspection of these principles will be carried out in a sequel paper.

To indicate another useful outreach of the density notion we discuss finally an application to the standard Ramsey function R_2^2 . The underlying vision is to indicate that there are quite a few interesting connections between finitary and infinitary Ramsey theory which are not fully explored yet.

The paper grew out of discussions held at a workshop in Utrecht University in May 2005. The workshop was sponsored by Netherlands Organization for Scientific Research (NWO), grant number 613.080.000.

Now, let us list a few definitions which will be used throughout the article. Let $M \models I\Delta_0$ be nonstandard.

- (1) An initial segment $I \subseteq M$ is a cut if I is closed under successor.
- (2) A cut I is semi-regular if for every M -definable function F with $\text{dom } F = [0, a], a \in I$, $\text{range}(F) \cap I$ is bounded in I . Alternatively, we can define I to be semi-regular if for every M -finite set S , if $\text{card}(S) \in I$ then $S \cap I$ is bounded in I . Every semi-regular cut satisfies WKL_0 (Lemma IX.3.11 in [10]).
- (3) A cut I is regular if for every M -definable partition $\langle A_i \mid i < a \in I \rangle$ of a set $A \supseteq I$, there is $i < a$ such that A_i is I -unbounded. Every regular cut satisfies $B\Sigma_2$ (see [6], page 216).
- (4) A cut I is strong if for every $n \in \mathbb{N}$, for every partition P of $[I]^n$ into $a \in I$ parts, there is an I -unbounded P -homogeneous subset of I . J. Paris and L. Kirby proved in [6] that for semi-regular cuts I , I is strong if and only if $I \rightarrow (I)_2^3$ if and only if there is an elementary end-extension $K \succ M$ such that there is a point of K between I and $M \setminus I$ and the same subsets of I are coded in K as in M . Every strong initial segment satisfies PA (see [6], page 223).
- (5) We say that a function (colouring, partition) $f: [\mathbb{N}]^2 \rightarrow \mathbb{N}$ is regressive if for all $x < y$ in \mathbb{N} , $f(x, y) \leq x$. Regressive Ramsey Theorem for pairs, RegRT^2 is the following statement in the language of second-order arithmetic: for any regressive function $f: [\mathbb{N}]^2 \rightarrow \mathbb{N}$, there is an infinite subset $X \subseteq \mathbb{N}$ such that for all $x < y < z$ in X , $f(x, y) = f(x, z)$, i.e., the value of f on X depends only on the first coordinate.
- (6) Canonical Ramsey Theorem for pairs, CRT^2 , is the following statement in the language of second-order arithmetic: for any $f: [\mathbb{N}]^2 \rightarrow \mathbb{N}$, there is an infinite $X \subseteq \mathbb{N}$ such that one of the four situations occur:
 - (a) $f|_{[X]^2}$ is injective;
 - (b) $f|_{[X]^2}$ is constant (homogeneity);

- (c) $f|_{[X]^2}$ depends only on the first coordinate (min-homogeneity);
 - (d) $f|_{[X]^2}$ depends only on the second coordinate (max-homogeneity).
- The set X is called canonical for f .

1 Ramsey Theorem for pairs and two colours

We shall consider a first-order theory $T = I\Sigma_1 + \bigcup_{n \in \omega} \forall a \exists b [a, b]$ is n -dense(2, 2). It is clear that $T \subset \text{WKL}_0 + \text{RT}_2^2$ because for every concrete $n \in \omega$, the statement $\forall a \exists b [a, b]$ is n -dense(2, 2) can be proved by n applications of RT_2^2 .

The aim of this section is to show that actually T achieves a lot of the strength of $\text{WKL}_0 + \text{RT}_2^2$: it is equiconsistent with $\text{WKL}_0 + \text{RT}_2^2$ and, moreover, already captures all Π_2 -consequences of $\text{WKL}_0 + \text{RT}_2^2$ and thus has the same provably recursive functions.

Theorem 1. *The set of Π_2 -consequences of $\text{WKL}_0 + \text{RT}_2^2$ coincides with the set of Π_2 -consequences of T .*

Proof.

Suppose $\Phi = \forall x \exists y \varphi(x, y)$ is such that $T + \neg\Phi$ is consistent. Let us show that $\text{WKL}_0 + \text{RT}_2^2 + \neg\Phi$ is also consistent.

Let $M \models T + \neg\Phi$ be countable and nonstandard, $a \in M$ be such that $M \models \forall y \neg\varphi(a, y)$. Notice that for each $n \in \mathbb{N}$,

$$M \models \exists z > a ([a, z] \text{ is } n\text{-dense}(2, 2)).$$

By Σ_1 -overspill, there is $b > a$ and $d > \mathbb{N}$ such that

$$M \models [a, b] \text{ is } d\text{-dense}(2, 2).$$

Let us define a cut I in M such that $a \in I < b$ and $I \models \text{WKL}_0 + \text{RT}_2^2$. Since $I \subset M$, all Π_1 truths are inherited from M , thus $I \models \neg\Phi$. Since M is countable, we can enumerate all M -coded 2-colourings of pairs of $[a, b]$ as $\{P_k : [a, b]^2 \rightarrow 2 \mid k \in \omega, k \geq 1\}$. Also, enumerate all M -finite sets as $\{S_i \mid i \in \omega\}$ so that each set appears infinitely-many times.

Define a sequence of sets $[a, b] = X_0 \supseteq Y_0 \supseteq X_1 \supseteq Y_1 \supseteq \dots$ such that for every $i \in \omega$,

$$\begin{aligned} X_i &\text{ is } (d - 2i)\text{-dense}(2, 2), \\ X_i &\text{ is homogeneous for } P_i, \\ Y_i &\text{ is } (d - 2i - 1)\text{-dense}(2, 2), \end{aligned}$$

if $\text{card } S_i \geq \min X_i - 1$ then $Y_i = X_i$, otherwise we shall find $u, v \in X_i$ such that

$$(u, v) \cap X_i \text{ is } (d - 2i - 1)\text{-dense}(2, 2) \\ \text{and } (u, v) \cap S_i = \emptyset$$

and we define $Y_i = (u, v) \cap X_i$. Let us show that indeed at every stage $i \in \omega$, the points u and v can be found. Define $P: [a, b]^2 \rightarrow 2$ as follows.

$$P(x, y) = \begin{cases} 1 & \text{if } \exists s \in S_i \quad x \leq s < y \\ 0 & \text{otherwise} \end{cases}$$

Let Y_i be P -homogeneous and $(d - 2i - 1)$ -dense(2, 2). If $P|_{[Y_i]^2} = 1$ then there is at most one element of Y_i between any two neighboring points of S_i , hence $\text{card}(Y_i) \leq \text{card}(S_i) + 1$. However,

$$\min Y_i < \text{card}(Y_i) \leq \text{card}(S_i) + 1 < \min X_i \leq \min Y_i,$$

a contradiction. Hence $P|_{[Y_i]^2} = 0$ and indeed $Y_i \subseteq (u, v) \cap X_i$ for some u and v . Without loss of generality we may assume that $Y_i \subseteq (u, v) \cap X_i$.

Define our initial segment $I = \sup_{i \in \omega} \min X_i$ and a collection of sets $\mathcal{X} = \{A \subseteq I \mid A = B \cap I \text{ for some } M\text{-finite set } B\}$. To check that I is semi-regular (which yields WKL_0 by a standard result from Simpson's book [10]), consider any M -set S such that $\text{card}(S) \in I$. There is a large enough index k such that

$$\text{card}(S) < \min X_k - 1$$

$$\text{and } S = S_k.$$

Notice that, by our construction, $(\min Y_k, \max Y_k) \cap S_k = \emptyset$, hence $S \cap I$ is bounded in I by $\min Y_k$. Hence $(I, \mathcal{X}) \models \text{WKL}_0$.

To show that $I \models \text{RT}_2^2$, first notice that for every $i \in \omega$, X_i is unbounded in I because X_i contains the set $\{\min X_k \mid k > i\}$, which is unbounded in I . Let $P: [I]^2 \rightarrow 2$ be an arbitrary coloring in (I, \mathcal{X}) . There is $k \in \omega$ such that $P = P_k \cap I$. Notice that X_k is P_k -homogeneous. Hence $X_k \cap I$ is a P -homogeneous unbounded set in I .

Thus $(M, \mathcal{X}) \models \text{WKL}_0 + \text{RT}_2^2 + \neg\Phi$. □

To indicate the strength of the notion n -dense(2,2) let us observe that PRA does not prove $\forall n \forall a \exists b [a, b]$ is n -dense(2,2). For, let $F_0(k) = k + 1$ and $F_{m+1}(k) = F_m^k(k)$ so that $k \mapsto F_k(k)$ is Ackermannian.

Proposition 2. *If $Z \subseteq [k + 1, F_{n+1}(k)]$ then Z is not n -dense(2,2).*

Proof.

By induction on n . First we treat the induction step. Define $\chi: [k+1, F_n^k(k)]^2 \rightarrow$

2 as follows: $\chi(x, y) = 1$ if there is a $z < k$ such that $x, y \in [F_n^z(k) + 1, F_n^{z+1}(k)]$ and put $\chi(x, y) = 0$ otherwise. Assume that $Y \subset Z$ is χ -monochromatic. If the colour of Y is 0 then $\text{card}(Y) \leq k < k + 1 \leq \min(Y)$. If the colour of Y is 1 then $Y \subseteq [F_n^z(k) + 1, F_n^{z+1}(k)] = [F_n^z(k) + 1, F_n(F_n^z(k))]$. By induction hypothesis Y is not $(n - 1)$ -dense(2, 2). Now assume that $n = 1$. Then $[k + 1, F_1^k(k)] = [k + 1, 2k]$ cannot be a 1-dense(2, 2) set. For if $Z \subseteq [k + 1, 2k]$ then $|Z| \leq k < k + 1 \leq \min(Z)$. \square

2 Regressive Ramsey Theorem for pairs

We say that X is 0-regressively dense(k) if $|X| \geq \max\{\min(X), 3\}$ and for every $n \in \omega$, X is $(n + 1)$ -regressively dense(k) if for every regressive partition $P: [X]^k \rightarrow \mathbb{N}$, there exists a P -min-homogeneous $Y \subseteq X$ which is n -regressively dense(k).

Theorem 3. *Let T be $I\Sigma_1 + \bigcup_{n \in \omega} \forall a \exists b [a, b]$ is n -regressively dense(2). Then $\text{WKL}_0 + \text{RegRT}^2$ has the same Π_2 -consequences as T .*

Proof. The setup is the same as in Theorem 1, so let $M \models \text{“}[a, b]$ is d -regressively dense(2)” for some $d > \mathbb{N}$. Let us enumerate all regressive M -coded partitions of $[a, b]^2$ as $\{P_k: [a, b]^2 \rightarrow [0, b] \mid k \in \omega, k \geq 1\}$ and all M -finite sets as $\{S_i \mid i \in \omega\}$ so that each set appears infinitely-many times. Define a sequence $[a, b] = X_0 \supseteq Y_0 \supseteq \dots \supseteq X_i \supseteq Y_i \supseteq \dots$ such that for every $i \in \omega$,

$$X_i \text{ is } (d - 2i) \text{ - regressively dense(2);}$$

$$X_i \text{ is min-homogeneous for } P_i;$$

$$Y_i \text{ is } (d - 2i - 1) \text{ - regressively dense(2);}$$

if $|S_i| > \min X_i - 2$ then $Y_i = X_i$, otherwise introduce

$$P(x, y) = \begin{cases} 0 & \text{if there is } s \in S_i \quad x \leq s < y \\ 1 & \text{otherwise} \end{cases}$$

Let Y_i be a P -min-homogeneous $(d - 2i - 1)$ -regressively dense(2) subset of X_i and consider the function $P(\min Y_i, z)$. If $P(\min Y_i, z) \equiv 0$ on Y_i then each interval in S_i contains at most one element of Y_i which clearly implies the contradiction: $\min Y_i < |Y_i| \leq |S_i| + 1 < \min X_i \leq \min Y_i$. Hence $P(\min Y_i, z) \equiv 1$ on Y_i and there are $u, v \in S_i$ such that $(u, v) \cap S_i = \emptyset$ and $Y_i \subseteq (u, v) \cap X_i$. Define $I = \sup_{i \in \omega} \min X_i$ and notice that by semi-regularity $I \models \text{WKL}_0$ and by our construction $I \models \text{RegRT}^2$. \square

We conclude this section with indicating that the 1-consistency of ACA_0 is equivalent with the assertion $\forall n \forall a \exists b [a, b]$ is n -regressively dense(2) over,

say, $I\Sigma_1$.

Notice that RegRT^2 is equivalent to ACA_0 over RCA_0 . This is proved, for example, in J. Hirst's thesis [5]. J. Hirst gave credits for this result to P. Clote. An alternative proof is provided by Proposition 5 of this paper. Thus

$$\forall n \forall a \exists b [a, b] \text{ is } n\text{-regressively dense(2)}$$

implies in particular the 1-consistency of PA.

Proposition 4. *Let $\Pi := \{\varphi \mid \varphi \in L_{\text{PA}} \cap \Pi_1 \text{ \& } \mathbb{N} \models \varphi\}$ Assume that*

$$M \models I\Sigma_1 + \forall n \forall a \exists b [a, b] \text{ is } n\text{-regressively dense(2)} + \Pi$$

is countable and nonstandard. Then there exists a cut $I \subseteq M$ such that

$$I \models \text{WKL}_0 + \text{RegRT}^2 + \Pi.$$

Proof. Choose n nonstandard in $M \models I\Sigma_1 + \Pi$. Let X_n be n -regressively dense(2). Choose an enumeration P_k of all M -coded regressive partitions with domain $[0, \max X_n]^2$ for standard numbers k . Choose recursively $(n - k - 1)$ -regressively dense(2) sets X_{n-k-1} which are min-homogeneous for P_k . Let

$$I := \{a \mid \exists k \text{ such that } a < \min(X_k)\}.$$

Then $X_k \cap I$ is unbounded in I for each k . Indeed, let $k \in \omega$ and $a \in I$. Assume that $a < \min(X_i)$ for some i . Define a partition $Q : [X_k]^2 \rightarrow \mathbb{N}$ as follows

$$Q(x, y) := \begin{cases} a & \text{if } a < x \\ 0 & \text{if } x \leq a. \end{cases} \quad (1)$$

Then Q is M coded. Choose $X_l \subseteq X_k \cap X_i$ min-homogeneous for Q . Then $a < \min(X_i) \leq \min(X_l) \leq |X_l|$. Hence the colour can not be 0 and we are done. In order to conclude that $M \models \text{WKL}_0$, we show that I is semi-regular. We shall conclude $I \models \text{RegRT}^2$ by construction and $I \models \Pi$ since $M \models \Pi$.

Let $a \in I$ and $F : a \rightarrow I$ be an M -coded function. We show that the range of F is bounded in I . Assume that the range of F is unbounded in I . Then we can define the following partition P .

$$P(x, y) := \begin{cases} \min i < a & \text{such that } F(i-1) \leq y < F(i) \text{ if } F(x) \geq y \text{ or } x \geq a \\ a & \text{if } F(x) < y \text{ and } x < a. \end{cases} \quad (2)$$

Let X_i be min-homogeneous for P . If the colour is 0 then the cardinality of X_i is bounded by $a < \min(X_i)$ since $X_i \setminus \max(X_i) \subseteq a$ as we will show now. If $x < y < z$ in X_i where $y, z > a$ then $P(y, z) = i$. But then the elements in X_i above y would be bounded by $F(i)$ in contradiction to the unboundedness of

X_i in I . If the colour is i then $F(i-1) \leq X_i < F(i)$. But $X_i \cap I$ is unbounded in I . Contradiction. \square

3 Characterization of strong cuts in terms of Regressive Ramsey Theorem for pairs

The following result (due to P. Clote) shows that the Regressive Ramsey Theorem implies ACA_0 . Related results can be found, for example, in [5,8].

Proposition 5. (*Characterization of strong initial segments*).

Let $M \models I\Delta_0 + \text{exp}$ be countable and nonstandard, $I \subset M$ be a semi-regular initial segment. Then I is strong if and only if $I \rightarrow (I)_{\text{regr}}^2$.

Proof. We shall use the following characterization of strong initial segments from [6]: for semi-regular $I \subset M$, I is strong if and only if there is $K \succ M$ such that

- (1) I is an initial segment of K ;
- (2) there is an element of K between I and $M \setminus I$;
- (3) K codes the same subsets of I as does M .

The proof will be similar to the proof of I is strong $\Leftrightarrow I \rightarrow (I)_2^3$ in [6].

Let $I \subset M$ be semi-regular and such that $I \rightarrow (I)_{\text{regr}}^2$. Let us first show that I is a regular initial segment: consider an M -coded partition $\{A_i \mid i < a \in I\}$ of I and let us show that for some $i < a$, A_i is I -unbounded. Consider a regressive coloring

$$P(x, y) = \begin{cases} i + 1 & \text{if } i < x \text{ and both } x \text{ and } y \text{ are in } A_i \\ 0 & \text{otherwise} \end{cases}$$

Choose a P -min-homogeneous, I -unbounded set $X \subset I$. If for some $y > x > a$ we have $P(x, y) = i \neq 0$ then A_i is I -unbounded since it contains $\{y \in X \mid y > x\}$. Otherwise define a function $F: \{x \in X \mid x > a\} \rightarrow a$ as $F(x) = i$ such that $x \in A_i$. Since $P \equiv 0$ on $\{x \in X \mid x > a\}$, F is injective and F^{-1} contradicts semi-regularity. Thus I is regular.

Enumerate all M -coded partitions of I as $\{\langle A_j^k \mid j < a_k \in I \rangle \mid k \in \omega\}$ and all regressive M -coded functions as $\{f_k: [I]^2 \rightarrow I\}$. Build a descending sequence of sets $I = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ such that for all $k \in \omega$,

- (1) X_k is an M -coded I -unbounded set;
- (2) X_{2k+1} is A_j^k for some $j < a_k$;
- (3) X_{2k+2} is f_k -min-homogeneous.

In the boolean algebra of all M -coded subsets of I , consider the ultrafilter \mathcal{U} of all sets that contain X_k for some $k \in \omega$. Define $K = \{f: I \rightarrow M \mid f \text{ is coded in } M\}$ with equality and operations on K defined modulo the ultra-filter \mathcal{U} . Clearly, $K \succ M$.

It is easy to see that I is an initial segment of K . Indeed, assume that an element $[f]$ slipped below $a \in I$. Then $\{x \in I \mid f(x) < a\} \in \mathcal{U}$. Define a partition as follows: $A_i = \{x \in I \mid f(x) = i\}$ for every $i < a$. By the construction of \mathcal{U} , for some $i < a$, $A_i \in \mathcal{U}$ hence $K \models [f] = i$.

Let us finally show that K codes the same subsets of I as M . Suppose $A \subset I$ is coded in K by a point $[f]$, $f: I \rightarrow M$. For every $i \in I$, let $A(i)$ be the set coded by $f(i)$. Define $g(x, y)$ as the code of $\{z < \log x \mid z \in A(y)\}$. Clearly, g is regressive. Consider a set $X \in \mathcal{U}$ that is min-homogeneous for g . Notice that for all $x < y < w$ in X ,

$$A(y) \cap \log x = A(w) \cap \log x.$$

That is, “whether z belongs or not belongs to A ” stabilizes after the first moment of X exceeding 2^z . Since I is semi-regular I is closed under exponentiation.

Now, remembering the formulas that define f and X in M , notice that A can be defined in M by the formula $z \in A \leftrightarrow$ “for all $x < y$ in X such that $2^z < x$, $z \in A(y)$ ”. \square

4 Canonical Ramsey Theorem for pairs

Definition 2. We say that X is 0-canonically dense if $|X| > \min X$. For every $n \in \omega$, X is $n+1$ -canonically dense if for every coloring f of $[X]^2$, there is a canonical n -canonically dense subset of X . The statement “ $\forall a \exists b [a, b]$ is n -canonically dense” will be denoted by n -candense(2).

Theorem 6. $\text{WKL}_0 + \text{CRT}^2$ is Π_2 -conservative over $I\Sigma_1 + \bigcup_{n \in \omega} n$ -candense(2).

Proof. The proof repeats the proof of Theorem 1 with one exception, the definition of Y_i . Let us consider the set S_i such that $\text{card } S_i < \min X_i - 1$. Consider the coloring

$$P(x, y) = \begin{cases} 0 & \text{if } \exists s \in S_i \ x \leq s < y \\ x - 1 & \text{otherwise} \end{cases}$$

Choose Y_i as a $(d - 2i - 1)$ -candense(2) subset of X_i , canonical for P . If $P|_{[Y_i]^2}$ is injective then clearly $(\min Y_i, \max Y_i) \cap S_i = \emptyset$ as we wish. If $P|_{[Y_i]^2}$ is

constant, not 0, then for any three different $x < y < z$ in Y_i ,

$$x - 1 = P(x, y) = P(y, z) = y - 1,$$

a contradiction. Hence $P|_{[Y_i]^2} \equiv 0$ and we notice that

$$\min Y_i < |Y_i| \leq |S_i| + 1 < \min X_i \leq \min Y_i,$$

a contradiction.

If Y_i is P -min-homogeneous then consider as a function of one argument $P(\min Y_i, y)$, which is constant on $[Y_i]^2$. If $P(\min Y_i, y) \equiv \min Y_i - 1$ then $(\min Y_i, \max Y_i) \cap S_i = \emptyset$ as we wish. Otherwise every interval in S_i contains at most one element of Y_i and (apart from the last interval that intersects with Y_i which may contain two elements of Y_i). Again we conclude

$$\min Y_i < |Y_i| \leq |S_i| + 1 < \min X_i \leq \min Y_i,$$

which is a contradiction.

If Y_i is P -max-homogeneous then consider the function $P(x, \max Y_i)$, which is constant on Y_i . Clearly $P(x, \max Y_i) \equiv 0$ and again $(\min Y_i, \max Y_i) \cap S_i = \emptyset$ as we wanted. \square

5 Characterization of strong cuts in terms of Canonical Ramsey Theorem

Finally we give a characterization of ACA_0 in terms of the Canonical Ramsey theorem. Related results can be found, for example, in Mileti's thesis [8].

Proposition 7. *(A characterization of strong cuts in terms of Canonical Ramsey closedness). Let $M \models I\Delta_0 + \text{exp}$ be countable and nonstandard, $I \subset M$ be a semi-regular initial segment. Then I is strong if and only if for every M -coded coloring $f: [I]^2 \rightarrow I$, there is an I -unbounded f -canonical subset of I .*

Again, notice that this observation implies that over RCA_0 , the set of first-order consequences of CRT^2 coincides with PA.

Proof. Suppose that for every M -coded coloring $f: [I]^2 \rightarrow I$, there is an I -unbounded f -canonical subset of I . Let us show that I is strong.

Again let us start off by showing that I is regular. Consider an M -coded partition $\{A_i \mid i < a \in I\}$ of I and let us show that for some $i < a$, A_i is

unbounded in I . Define a coloring

$$P(x, y) = \begin{cases} i + 1 & \text{if } i < a \text{ and } x, y \in A_i \\ 0 & \text{otherwise} \end{cases}$$

Consider a P -canonical I -unbounded subset $X \subseteq I$.

If X is P -monochromatic and $P|_{[X]^2} \equiv i + 1$ then $X \subseteq A_i$, hence A_i is I -unbounded. If $P|_{[X]^2} \equiv 0$ then define a function $f: a \rightarrow I$ as $f(i) =$ the first $x \in X$ such that $x \in A_i$ if exists and 0 otherwise. The function f is unbounded in I contradicting semi-regularity.

If X is min-homogeneous, see the proof above for RegRT^2 .

If X is max-homogeneous define $Q: X \rightarrow a$ as $Q(y) = P(x, y)$. Notice that for every $y \in X$, either all smaller $x \in X$ are in the same A_i with y or all are not. If $Q(y)$ takes the value 0 unboundedly-often then clearly Q^{-1} is unbounded in I contradicting semi-regularity. Otherwise notice that from some moment in I , $Q \equiv i + 1$. Indeed, consider $x < y < z$ in X . Since $P(x, z) = P(y, z)$, we conclude that x and y are in the same A_i as z for some i . This A_i is the unbounded set we seek.

If P is injective on $[X]^2$ then consider $Q(x) = P(\min X, x)$ and notice that Q^{-1} is an I -unbounded function defined on a subset of the segment $[0, a]$, contradicting semi-regularity. Hence I is regular.

Again, as in Proposition 5, enumerate all M -coded partitions of I as $\{\langle A_i^k \mid i < a_k \rangle, k \in \omega\}$ and all M coded functions as $\{f_k: [I]^2 \rightarrow I\}_{k \in \omega}$ and build a descending sequence $I = X_0 \supseteq X_1 \supseteq \dots$ of M -coded I -unbounded sets such that:

- (1) X_{2k+1} is A_j^k for some $j < a_k$;
- (2) X_{2k+2} is f_k -canonical.

Define the ultrafilter \mathcal{U} as the set of all M -coded subsets of I that contain X_k for some $k \in \omega$. Define $K = \{f: I \rightarrow M \mid f \text{ is coded in } M\}/\mathcal{U}$ and notice as before that I is an initial segment of M .

Let us show that K codes the same subsets of I as M . Suppose $A \subset I$ is coded in K by a point $[f]$, $f: I \rightarrow M$. For every $i \in I$, let $A(i)$ be the subset coded by $f(i)$. Define $g(x, y)$ as $\{z < x \mid z \in A(y)\}$ and consider $X \in \mathcal{U}$, canonical for g . Let us show that then A is definable in M by the formula

$$z \in A \leftrightarrow \text{for all } x, y \in X, \text{ if } z < x < y \text{ then } z \in A(y).$$

Indeed, if X is g -homogeneous then there is an M -definable set $B \subset [0, \min X]$

such that for all $y \in X$, $y > \min X$, $A(y) = B$.

If X is min-homogeneous for g then clearly for all $x < y < w$ in X , $A(y) \cap [0, x) = A(w) \cap [0, x)$ and our formula defines A .

If X is max-homogeneous for g then for any $x_1 < x_2 < y$ in X , $\{z < x_1 \mid z \in A(y)\} = \{z < x_2 \mid z \in A(y)\}$, hence A is contained in $[0, \min X)$ and is defined by our formula.

Let us show that g can not be injective on X . Put $x_i =$ the i th element of X and consider $Q: I \rightarrow \min X$ defined as $Q(i) = g(\min X, x_i)$. By semi-regularity, I is closed under exponentiation, hence there are more than $2^{\min X}$ elements in X , contradiction with Q being injective. Hence I is strong. \square

6 The infinitary Erdős-Moser result

A tournament is a complete directed simple graph. Let CAC be the statement that every infinite partial order has an infinite chain or antichain and let EM be the statement that every infinite tournament contains an infinite transitive subtournament. (A finitary analogue of EM was investigated by Erdős and Moser.) Then RT_2^2 proves CAC and EM. But we also have a reversal.

Theorem 8. $\text{RCA}^* + \text{EM} + \text{CAC} \vdash \text{RT}_2^2$.

Proof. Let $P: [\mathbb{N}]^2 \rightarrow 2$ be given. Define $x \leftarrow y$ if and only if either $x < y$ and $P(x, y) = 0$ or $y < x$ and $P(x, y) = 1$.

Then $\langle \mathbb{N}, \leftarrow \rangle$ is an infinite tournament. By EM choose an infinite $S \subset \mathbb{N}$ such that $\langle S, \leftarrow \rangle$ is an infinite transitive subtournament hence a linear ordering. Now put $x \prec y$ if and only if $x \leftarrow y$ and $P(x, y) = 0$. Then $\langle S, \prec \rangle$ is an infinite partial order. With CAC choose $H \subseteq S$ infinite which is either a chain or an antichain. If $\langle H, \prec \rangle$ is a chain then for all $x, y \in H$ with $x < y$ we have $P(x, y) = 0$. If $\langle H, \prec \rangle$ is an antichain then for all $x, y \in H$ with $x < y$ we have $P(x, y) = 1$. Hence H is P -homogeneous. \square

(Antonio Montalban informed us kindly that RT_2^2 even follows from $\text{ASDS} + \text{EM}$.) Now we show that EM can not be proved within RCA_0 .

Theorem 9. *There exists a recursive tournament without an infinite recursive transitive subtournament.*

Proof. Let a recursive partition $P: [\mathbb{N}]^2 \rightarrow 2$ be given such that every infinite P -homogeneous set is not Σ_2^0 . Define $x \rightarrow y$ if and only if either $x < y$ and $P(x, y) = 0$ or $y < x$ and $P(x, y) = 1$.

Then $\langle \mathbb{N}, \rightarrow \rangle$ is an infinite recursive tournament. By EM choose an infinite $S \subseteq \mathbb{N}$ such that $\langle S, \rightarrow \rangle$ is an infinite transitive subtournament hence a linear ordering.

Then by Theorem 3.8 of [3] (Herrmann's theorem) we find a sequence s_i in S which is recursive and increasing or Δ_2^0 and decreasing. We may assume that for $i < j$ we have $s_i < s_j$ both with respect to the standard ordering on \mathbb{N} . Let H be the range of $i \mapsto s_i$. Then H is Σ_2^0 . If s_i was increasing with respect to \rightarrow then for $x, y \in H$ with $x < y$ we have $P(x, y) = 0$ and H is homogeneous. If s_i was decreasing with respect to \rightarrow then for $x, y \in H$ with $x < y$ we have $P(x, y) = 1$ and H is homogeneous. But P does not have a homogeneous H which is in Σ_2^0 . \square

Here is the density notion for tournaments. We call $X \subseteq \mathbb{N}$ 0-EM-dense if $|X| \geq \min X$. We say that X is $(n + 1)$ -EM-dense if for all R such that $\langle X, R \rangle$ is a tournament there exists $Y \subseteq X$ such that $\langle Y, R \rangle$ is a transitive subtournament and such that Y is n -EM-dense.

Theorem 10. *Suppose that $|X| \geq 2_n(m)$ and $\min X = m$. Then X is n -EM-dense.*

Proof. For $n = 0$ the assertion follows from the assumption that $\min X = m$. Now, assuming that the assertion holds for n , prove it for $n + 1$. Consider a set X such that $\min X = m$ and $|X| \geq 2_{n+1}(m) = 2^{2^n(m)}$. Pick an arbitrary relation R such that $\langle X, R \rangle$ is a tournament. Let $I(m) := \{y \mid \langle y, m \rangle \in R\}$ and $O(m) := \{y \mid \langle m, y \rangle \in R\}$. One of these sets has cardinality at least $2^{2^n(m)-1}$. Without loss of generality, assume that this is $I(m)$. Now by iteration $I(m)$ has a transitive subtournament $\langle Y, R \rangle$ of cardinality $2_n(m) - 1$. Hence $\langle \{m\} \cup Y, R \rangle$ is a transitive subtournament of cardinality $2_n(m)$. Moreover $\{m\} \cup Y$ has minimum m . By induction hypothesis, $\{m\} \cup Y$ is n -EM-dense. Therefore X is $(n + 1)$ -EM-dense. \square

So, somewhat surprisingly, the theory $I\Sigma_1 + \bigcup_{n \in \mathbb{N}} (\forall a)(\exists b)([a, b] \text{ is } n\text{-EM-dense})$ collapses into $I\Sigma_1$ indicating a certain weakness of EM.

7 A phase transition for R_2^2

Till now we have used densities as principles to approach combinatorial principles from below. In this small section we shall indicate that densities also have some outreach to finite Ramsey theory thereby confirming that there are quite a few connections between finitary and infinitary Ramsey theory. Concerning this let us also remark that we have been originally led to the principle EM by finite Ramsey theory.

Let us consider the Ramsey function for pairs. We are interested in a phase transition in terms of densities. If f is a number theoretic function then we call a finite set X of natural numbers 0 -dense(f, k, l) if $\text{card } X \geq \max \{3, f(\min(X))\}$. We say that X is $(n + 1)$ -dense(f, k, l) if for any $F: [X]^k \rightarrow l$ there exists a $Y \subseteq X$ such that $F|_{[Y]^2}$ is constant and Y is n -dense(f, k, l).

Recall that $R_2^2(k)$ is the least m such that for every $F: [m]^2 \rightarrow 2$ there exists a monochromatic $Y \subseteq m$ such that $\text{card}(Y) \geq k$. Then, by Erdős's probabilistic method, we know the classical lower bound $R_2^2(k) \geq 2^{\frac{k}{2}}$ for all k [4]. Elementary combinatorics yields further the well known upper bound $R_2^2(k) \leq 2^{2 \cdot k}$ for all k [4]. It is open (an Erdős USD 100 problem) whether the limit $\lim_{k \rightarrow \infty} (R_2^2(k))^{\frac{1}{k}}$ exists. (Determining the value is an Erdős USD 250 problem.)

Let \log_4 denote the logarithm with respect to base 4. Then the following result holds [11].

Theorem 11. *Let $f(i) := \lceil \log_4(\log_4(i - \frac{1}{2})) \rceil$. Assume that*

$$\rho := \lim_{k \rightarrow \infty} (R_2^2(k))^{\frac{1}{k}}$$

exists. Let

$$\mu := \inf \{r \in \mathbb{R} \mid (\exists K)(\forall m \geq K) [4^{4^m}, 4^{4^m} + \lfloor (r)^{\lfloor (r)^m \rfloor} \rfloor] \text{ is } 2\text{-dense}(f, 2, 2)\}.$$

Then $\rho = \mu$.

References

- [1] Friedman, H., McAloon, K., Simpson, S. (1982). A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis. *Patras Logic Symposium* (Patras, 1980), pp. 197–230.
- [2] Cholak, P., Jockusch, C., Slaman, T. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic* 66 (2001), no. 1, 1–55.
- [3] Cholak, P., Marcone, A., Solomon, R. (2004). Reverse mathematics and the equivalence of definitions for well and better quasi-orders, *Journal of Symbolic Logic* 69, pp 683-712.
- [4] Graham, R.L., Rothschild, B.L., Spencer, J.H. (1990). *Ramsey theory*. Wiley.
- [5] Hirst, J. (1987). *Combinatorics in subsystems of second order arithmetic*, Ph.D. Thesis, Pennsylvania State University.
- [6] Kirby, L., Paris, J. (1977). Initial segments of models of Peano's axioms. In: *Set theory and hierarchy theory, V (Proc. Third Conf., Bierutowice, 1976)*, pp. 211–226. *Lecture Notes in Math.*, Vol. 619. Springer, Berlin.

- [7] McAloon, K. (1985). Paris-Harrington incompleteness and progressions of theories. *Proceedings of Symposia in Pure Mathematics*, 42, pp. 447-460.
- [8] Mileti, J. (2004). Partition Theorems and Computability Theory. PhD Thesis, University of Illinois, Urbana-Champaign.
- [9] Paris, J. (1978). Some independence results for Peano arithmetic. *Journal of Symbolic Logic*, 43, pp. 725-731.
- [10] Simpson, S. (1997). *Subsystems of Second Order Arithmetic*. Springer.
- [11] Weiermann, A, (2006) *Phase transition thresholds for some natural subclasses of the computable functions*. Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, June 30-July 5, 2006, Proceedings. Lecture Notes in Computer Science 3988, Springer, 556-570.

ANDREY BOVYKIN
DEPARTMENT OF MATHEMATICS
BRISTOL UNIVERSITY, BS8 1TW, UNITED KINGDOM andrey@logic.pdmi.ras.ru

ANDREAS WEIERMANN
VAKGROEP ZUIVERE WISKUNDE EN COMPUTERALGEBRA
GHENT UNIVERSITY
KRIJGSLAAN 281 - GEBOUW S22
B9000 GENT, BELGIUM
weiermann@cage.ugent.be