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Several proofs of PA-unprovability

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Abstract. In this article we give unprovability proofs of several combinatorial statements using the method of indiscernibles. Our first two statements are strong versions of KM (the Kanamori-McAloon principle) and $\mathrm{KM}^{(2)}$, (the Kanamori-McAloon principle for pairs) that both allow unexpectedly simple proofs. We proceed to discuss results by A.Weiermann and G.Lee on $I\Sigma_k$-provability/unprovability of the principles $\mathrm{PH}^{(k+1)}_{\log^{(n)}}$ and $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ for different $n \in \mathbb{N}$. We contrast a result by G.Lee on provability of $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ for $n \geq k$ with the following result. If the condition "for all x, y in H, $x < y$ implies $2^x < y$" is imposed on the min-homogeneous set H then the modified statement $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ becomes $I\Sigma_k$-unprovable for all $n, k \in \mathbb{N}$. For $k \geq 2$, the restriction $|H| > 2^c$, where c is the second element of H also makes the modified $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ unprovable in $I\Sigma_k$ for all $n \in \mathbb{N}$.

The article is directed at a broad audience and is intended to be suitable for expository purposes.

Here, we present several model-theoretic proofs of unprovability (in Peano Arithmetic, PA, and in its fragments $I\Sigma_k$, the theories of induction for formulas containing not more than k quantifiers) usinig the method of indiscernibles. We give full proofs and intend this article to be suitable for expository purposes and accessible to mathematicians interested in "how is it possible to prove that a concrete statement about natural numbers is unprovable?".

1. Introduction

Peano Arithmetic (PA) is the first-order theory in the language

$$\mathcal{L} = \{+, \times, <, 0, 1\}$$

consisting of the following axioms: associativity and commutativity of $+$ and $\times$, the neutral elements are 0 and 1 respectively, distributivity, discrete linear order

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axioms for $<$, 1 is the successor of 0, $x < y \to x + z < y + z$, and the induction scheme: for every $\mathcal{L}$-formula $\varphi(x, \overline{y})$, we have an axiom

$$\forall \overline{y}[\ \varphi(0, \overline{y}) \wedge \forall x(\varphi(x, \overline{y}) \to \varphi(x + 1, \overline{y})) \to \forall x \varphi(x, \overline{y})\ ].$$

For $n \in \mathbb{N}$, define a Σ_n-formula as an $\mathcal{L}$-formula of the form

$$\exists x_1 \forall x_2 \exists x_3 \ldots \varphi(x, x_1, x_2, \ldots, x_n, \overline{y}),$$

where φ does not contain unbounded quantifiers, i.e., a Σ_n-formula is a formula (possibly with free variables) that starts with an existential quantifier and contains not more than n quantifiers altogether.

If we restrict the induction scheme to Σ_n-formulas then the theory obtained is denoted by $I\Sigma_n$, "induction for Σ_n-formulas". Clearly, for every $n \in \mathbb{N}$, $I\Sigma_n \subseteq I\Sigma_{n+1}$ and $\mathrm{PA} = \bigcup_{n=1}^{\infty} I\Sigma_n$. Also, it is known and easily provable nowadays that for every $n \in \mathbb{N}$, $I\Sigma_n \neq I\Sigma_{n+1}$.

An alternative axiomatization of $I\Sigma_n$ uses the instance of the least-number principle

$$\forall \overline{y}[\ \exists x \varphi(x, \overline{y}) \to \exists z(\varphi(z, \overline{y}) \wedge \forall w < z \neg \varphi(w, \overline{y}))]$$

for every Σ_n-formula $\varphi(x, \overline{y})$.

It is common to identify theorems of PA with "finite mathematics", that is the world of mathematical theorems which can be formulated in $\mathcal{L}$ and whose proof does not require the use of any notion of "infinite set" in an essential way. As typical arbitrary old examples of such "theorems of finite mathematics" I would quote:

(1) if $\pi(x)$ is the number of primes less than x then $\pi(x) \sim \int\limits_2^x \frac{dt}{\ln t}$, $x \to \infty$ (in order to define the notion of an intergral in the language of arithmetic we should only use Darboux partitions with rational ends; the notion of a limit is arithmetized easily, also, all necessary instances of theorems of complex analysis can be conducted in Peano Arithmetic);

(2) for every concrete continuous function $f\colon [a, b) \to \mathbb{R}$ definable in the language of arithmetic (this includes all functions working mathematicians encounter): $\frac{d}{dx}\left(\int\limits_a^x f(t)dt\right) = f(x)$;

(3) for every concrete $n \times n$ matrix $A(t)$ continuous on $[a, b]$, if $\Phi(t)$ is a matrix whose columns are solutions of $\frac{dx}{dt} = A(t)x$ then $W(t) = \det \Phi(t)$ can be calculated as $W(t) = W(a) \cdot \exp(\int\limits_a^t TrA(s)ds)$.

It is then an easy excercise to check that the usual proofs of the above theorems can be conducted in Peano Arithmetic.

However, it is widely believed that all $\mathcal{L}$-theorems of existing mathematics (apart from logicians' discoveries we shall be talking about below) can be proved even in $I\Sigma_2$. It would be a big surprise and an interesting result if someone managed to find an existing theorem in mathematics that can be formulated in $\mathcal{L}$ but does not have a proof formalizable in $I\Sigma_2$.

For a very long time it was believed that PA comprises an axiomatization of the set of all *'truths'* about natural numbers and finite sets until in 1931 K.Gödel proved that for every consistent recursive theory T containing PA, there is an $\mathcal{L}$-formula φ such that neither φ nor $\neg\varphi$ can be derived in T. An example of a sentence that is neither provable nor refutable in T is Con_T, the arithmetical formula expressing consistency of T ("for every natural number n, n is not a code of a sequence of formulas ending with $\exists x\ x \neq x$ and such that every formula in the sequence is either an axiom of T or is obtained from the previous ones by a rule of inference of the predicate calculus").

The first PA-unprovable statements of 'mathematical' character (i.e., not refering to arithmetization of syntax and provability) appeared in 1976 in the work of J.Paris (building upon joint work with L.Kirby [**19**]) and led to formulation of the Paris-Harrington Principle [**33**]:

$$\mathrm{PH} \ \leftrightarrow\ \forall mnc\exists N \left[\begin{array}{c}\text{for every } f\colon [N]^m \to c, \text{ there is an } f\text{-homogeneous}\\ H \subseteq N, |H| \geq n \text{ such that } |H| > \min H\end{array}\right].$$

PH is not provable in Peano Arithmetic. Moreover, for every $k \in \mathbb{N}$, the statement $\mathrm{PH}^{(k+1)}$ defined as

$$\mathrm{PH}^{(k+1)} \ \leftrightarrow\ \forall nc\exists N \left[\begin{array}{c}\text{for every } f\colon [N]^{k+1} \to c, \text{ there is an } f\text{-homogeneous}\\ H \subseteq N, |H| \geq n \text{ such that } |H| > \min H\end{array}\right]$$

is $I\Sigma_k$-unprovable [**35**] and is equivalent to $\mathrm{RFN}_{\Sigma_1}(I\Sigma_k)$, the 1-consistency of $I\Sigma_k$:

$$\forall \varphi \in \Sigma_1\ (\mathrm{Pr}_{I\Sigma_k}(\varphi) \to \varphi).$$

It says "for all Σ_1-statements φ, if $I\Sigma_k$ proves φ then φ holds". Unprovability of $\mathrm{RFN}_{\Sigma_1}(I\Sigma_k)$ in $I\Sigma_k$ easily follows from Gödel's Theorem: put φ to be $\exists x\ x \neq x$ to observe that $\mathrm{RFN}_{\Sigma_1}(I\Sigma_k)$ implies $\mathrm{Con}_{I\Sigma_k}$. Many statements equivalent to PH have been studied since: the Hercules-Hydra battle and termination of Goodstein sequences by L.Kirby and J.Paris [**36**], the flipping principle of L.Kirby [**20**], the arboreal statement by G.Mills [**31**], Pudlák's Principle [**12**], the kiralic and regal principles by P.Clote and K.McAloon [**6**].

An important PA-unprovable statement was introduced in [**16**] by A.Kanamori and K.McAloon. A function f in m arguments is called regressive if

$$f(x_0, x_1, \ldots, x_{m-1}) < x_0 \text{ for all } x_0 < x_1 < \cdots < x_{m-1}.$$

For regressive functions of m arguments, we cannot guarantee existence of a homogeneous set of size $(m+1)$, e.g., for $f(x_0, x_1, \ldots, x_{m-1}) = x_0 - 1$, every set of size $(m+1)$ is not homogeneous. However, we can talk about min-homogeneous sets: a set H is called min-homogeneous if for all $c_0 < c_1 < \cdots < c_{m-1}$ and $c_0 < d_1 < \cdots < d_{m-1}$ in H, $f(c_0, c_1, \ldots, c_{m-1}) = f(c_0, d_1, \ldots, d_{m-1})$. Now, KM is defined as:

$$\mathrm{KM} \leftrightarrow \forall man\exists b \left[\begin{array}{c}\text{for every regressive function } f \text{ defined}\\ \text{on } [a,b]^m, \text{ there is a min-homogeneous}\\ \text{set } H \subseteq [a,b] \text{ of size at least } n\end{array}\right].$$

The statement KM is unprovable in PA and is equivalent to PH. Also, $\mathrm{KM}^{(k)}$, the version of KM restricted to k-tuples, is equivalent to $\mathrm{PH}^{(k)}$.

The historical prototypes of the Paris-Harrigton Principle and the earlier PA-unprovable statements [**34**] are large cardinal axioms (for an early discussion of this connection, see [**18**]). In the case of arithmetic, the closedness properties postulated by large cardinal axioms correspond to closedness properties of initial segments of models of arithmetic. So far, the idea to look closely at arithmetical versions of different large cardinal axioms (and to go beyond J.Paris' semi-regularity, regularity, strength, extendibility and Ramseyness of initial segments [**19**], [**35**]) has not been really explored. We believe that it will eventually be very fruitful.

Very often an unprovable statement can be viewed as a 'miniaturization' of an infinitary theorem. A spectacular example of miniaturization is H.Friedman's Theorem [**42**] on unprovability of a finite version of the following theorem by J.Kruskal. Define a tree as a partially ordered set with the least element and such that the set of all predecessors of every point is linearly ordered. Then if $\{T_i\}_{i\in\mathbb{N}}$ is a countable sequence of finite trees then there are $i < j$ in $\mathbb{N}$ such that $T_i \hookrightarrow T_j$, i.e., there is an inf-preserving embedding from T_i into T_j. Friedman's Theorem says that

$$\forall k\ \exists N \left[\begin{array}{c} \text{if } \{T_i\}_{i=1}^{N} \text{ is a sequence of finite trees such} \\ \text{that for all } \ i \le N \text{ we have } |T_i| \le k+i \text{ then} \\ \text{there are } i, j \le N \text{ such that } i < j \text{ and } T_i \hookrightarrow T_j \end{array} \right]$$

is not provable in ATR_0, a theory stronger than Peano Arithmetic. It was later shown by M.Loebl and J.Matoušek [**26**] that if the condition $|T_i| \le k+i$ is replaced by $|T_i| \le k + \frac{1}{2}\log i$ then the statement becomes $I\Sigma_1$-provable but for the condition $|T_i| \le k + 4\log i$ the statement is PA-unprovable. What happens between $\frac{1}{2}$ and 4 was recently resolved by A.Weiermann [**46**]. Let α be the Otter's constant (the radius of convergence of $\sum_{i=0}^{\infty} t_i z^i$, where t_i is the number of finite trees of size i), $\alpha \approx 2.955765\ldots$. Then for any primitive recursive real number r,

1. if $r \le \frac{1}{\log\alpha}$ then the statement with the condition $|T_i| \le k + r\log i$ is $I\Sigma_1$-provable;
2. if $r > \frac{1}{\log\alpha}$ then it is PA-unprovable.

Another example is a theorem by H.Friedman, N.Robertson and P.Seymour on unprovability of the Graph Minor Theorem [**10**] (for graphs G and H, we say that H is a minor of G if H is obtained from G by a succession of three elementary operations: edge removal, edge contraction and removal of an isolated vertex):

$$\forall k\ \exists N \left[\begin{array}{c} \text{if } \{G_i\}_{i=1}^{N} \text{ is a sequence of finite graphs} \\ \text{such that for all } i \le N \text{ we have } |G_i| \le k+i \\ \text{then for some } i < j \le N,\ G_i \text{ is a minor of } G_j \end{array} \right]$$

is not provable in Π^1_1-CA_0, a very strong subsystem of the second-order arithmetic.

Many other examples of unprovable statements can be found in the *Contemporary Mathematics* volume 65 [**43**] devoted entirely to arithmetical unprovability results. Among the developments that escaped this volume I would like to mention K.McAloon's [**29**], [**30**] and Z.Ratajczyk's [**38**],[**39**] theories of iterations of the Paris-Harrington Principle, Cichon's treatment of Goodstein sequences [**5**], the treatment of PH by S.Kripke and S.Kochen [**22**] using ultraproducts, the Friedman-McAloon-Simpson early fundamental article [**8**] (a version of Galvin-Prikry partition theorem unprovable in ATR_0) and the subsequent article by S.Shelah [**44**], the Buchholz Hydra [**3**] and Friedman's early results on combinatorial statements unprovable in ZFC+ large cardinals [**9**], [**11**] (see also the article [**40**] by J.-P. Ressayre). Also, there is a whole range of recent results achieved by H.Friedman.

But of course this list is far from complete. The purpose of this introduction was to show some landmarks and to give an adequate impression of what a good theorem in this subject should look like.

Each independence proof of an arithmetical statement so far falls into one of the two categories:

(1) model-theoretic constructions showing how, assuming the statement, a model of a given theory can be built directly, "by hands";
(2) combinatorial proofs springing from the Ketonen-Solovay article [**17**] (showing combinatorially that the function arising from our statement eventually dominates every function of the Grzegorczyk-Wainer hierarchy (since all PA-provably recursive functions occur in this hierarchy, the result follows) or from the study of well-quasi-ordered sets [**42**]. Most of the proofs in [**43**] are of this category, as well as the articles [**27**] and [**28**] by M.Loebl and J. Nešetřil.

Apart from the original articles we mentioned above, other good sources reporting on proofs of category (1) would be the book [**13**] by P.Hájek and P.Pudlák and the papers [**1**], [**2**] by J.Avigad and R.Sommers (proof-theoretic aspects). A recent manuscript [**23**] by H.Kotlarski is an exposition of both approaches as well as of many different proofs of Gödel's Theorem.

Apparently, there is also a third category of unprovability proofs, proofs that interpret directly independence statements as reduction strategies for proof systems. We have little to say about it and refer the reader to a recent article by L.Carlucci [**4**] and the articles [**14**] and [**15**] by M.Hamano and M.Okada.

The proofs in this article are typical of category (1). We aimed to formulate simple statements whose proofs would be very short and rid of unnecessary combinatorial manipulations. The combinatorial statements discussed in this article can be shown to be provable in the second-order arithmetic, by an easy application of the Infinite Ramsey Theorem and we omit these proofs.

Also, as it often happens in this subject, all our statements imply 1-consistency of the theory we prove them to be independent of. We conjecture but do not prove that they are equivalent to its 1-consistency. So, apart from our model-theoretic proofs, direct combinatorial arguments are expected to be eventually found, showing our statements independent of $I\Sigma_k$ to be equivalent to $\mathrm{PH}^{(k+1)}$ or $\mathrm{KM}^{(k+1)}$ and the statements independent of PA to be equivalent to PH or KM.

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2. Monochromatic paths in oriented graphs

Let G_{ab} be the following oriented graph[1]: the vertices of G_{ab} are the natural numbers $a, a+1, \ldots, b$; there is an arrow from x to y if and only if $x < y$. The set of all arrows originating in x will be denoted by E_x. In G_{ab}, $|E_x| = b - x$.

Let G be an oriented graph and suppose a colouring χ of arrows of G is fixed. A path through G (a way along arrows) is called monochromatic with respect to χ if any two arrows originating in the same vertex of the path and finishing somewhere on the path (i.e., any two chords with the same origin) have the same colour.

THEOREM 1.
Let $A(a, b, n)$ be the following formula: "for any colouring χ of arrows of G_{ab} such that $|\chi(E_x)| \leq 2^{nx}$, there is a monochromatic path of length n". Define

$$\mathrm{MPG} \leftrightarrow \ \forall an \ \exists b \ A(a, b, n).$$

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[1] the words 'directed graph' ('digraph') and 'edges' are often used instead of 'oriented graph' and 'arrows'.

Then $I\Sigma_1 \nvdash$ MPG. Moreover, for every $a \in \mathbb{N}$, the function $F(a,n) = F_a(n) = \min c\ A(a, a+c, n)$ eventually dominates every primitive recursive function.

In the proof we shall use the fact that primitive recursive functions are provably recursive in $I\Sigma_1$. The converse (every $I\Sigma_1$-provably recursive function is primitive recursive) is also true and is a nontrivial theorem proved independently by G.Mints [**32**] and C.Parsons [**37**] (also G.Takeuti [**45**]). By $I\Delta_0$ we denote the theory $I\Sigma_0$ (as defined above), the theory of induction for formulas without unbounded quantifiers. The expression $x = \langle x_1, \ldots, x_n\rangle$ means that x is the code of the sequence $x_1, \ldots, x_n$, using some standard coding function. We shall build a model of $I\Sigma_1 + \neg$ MPG, thus showing that MPG cannot be a theorem of $I\Sigma_1$.

Proof.
Let $M \vDash I\Sigma_1$ be nonstandard, $a, b, n \in M$, $\mathbb{N} < n$. Assume that $M \vDash A(a,b,n) \wedge \forall y < b\ \neg A(a,y,n)$. Let $\psi_1(u,v), \ldots, \psi_n(u,v)$ be the first n Δ_0-formulas in the two free variables shown. Introduce the coloring: for the arrow xy, set

$$\chi(xy) = \left\langle \begin{array}{c} \{p < x \mid \psi_1(y,p)\} \\ \vdots \\ \{p < x \mid \psi_n(y,p)\} \end{array} \right\rangle .$$

Since for every $x \in [a,b]$, $|\chi(E_x)| \leq 2^{nx}$, we choose in G_{ab} a monochromatic path $C = \{c_i\}_{i=1}^{n}$. Notice that for any $c_{i_1} < c_{i_2} < c_{i_3}$ in C, the fact that $c_{i_1}c_{i_2}$ and $c_{i_1}c_{i_3}$ have the same colour implies that for any Δ_0-formula $\psi(u,v)$ and any $p < c_{i_1}$,

$$M \vDash \psi(c_{i_2}, p) \ \leftrightarrow\ \psi(c_{i_3}, p). \qquad (*)$$

Define the initial segment $I = \sup_{i\in\mathbb{N}} c_i$. It can easily be checked from $(*)$ that I is closed under addition and multiplication: for any $c_{i_1} < c_{i_2}$ in C, if for some $p < c_{i_1}$ we have $[\frac{c_{i_2}}{2}] = p$ then, by $(*)$, $[\frac{c_i}{2}] = p$ for all $i > i_1$. Hence all c_i for $i > i_1$ would coincide or differ by 1 which is not the case. Hence $2p < c_{i_2}$ and I is closed under addition. For multiplication, the proof is similar: if there was $p < c_{i_1}$ such that for all $i > i_1$, $[\sqrt{c_i}] = p$ then there would be at most $2p+1$ such elements of C, namely $p^2, p^2+1, \ldots, p^2+2p$. But this set can not accommodate infinitely-many elements of C since we already proved that the distance between them is greater than p.

In order to show that $I \vDash I\Sigma_1$, we check the least-element principle for an arbitrary Σ_1-formula. Consider a Σ_1-formula $\exists y\ \varphi(x,y,p)$ such that $I \vDash \exists y \varphi(e,y,p)$ for $\langle e,p\rangle < c_{i_1} \in I$. Now, considering the formula $\psi(u,v)$ defined as $\psi(u, \langle x,p\rangle) \leftrightarrow \exists y < u \varphi(x,y,p)$, and applying $(*)$ to it, we observe that

$$I \vDash \exists y \varphi(e,y,p) \Leftrightarrow M \vDash \exists y < c_{i_1+1} \varphi(e,y,p). \qquad (**)$$

Now, since $M \vDash I\Delta_0$, there is $x^* < c_{i_1}$ such that

$$M \vDash \exists y < c_{i_1+1}\ \varphi(x^*,y,p) \ \wedge\ \forall x < x^*\ \forall y < c_{i_1+1}\ \neg\varphi(x,y,p).$$

Hence, by $(**)$, $I \vDash \exists y \varphi(x^*,y,p) \ \wedge\ \forall x < x^*\ \forall y\ \neg\varphi(x,y,p)$ which means that $I \vDash I\Sigma_1$. Now, if it was the case that $I\Sigma_1 \vdash \forall an \exists y\ A(a,y,n)$ then there would exist $b' \in I$ such that $M \vDash A(a,b',n)$ contradicting the choice of b as the minimal one. Hence $I\Sigma_1 \nvdash \forall an\ \exists y\ A(a,y,n)$.

Let $a \in \mathbb{N}$. To show that $F_a(n)$ eventually dominates every primitive recursive function, consider an arbitrary Δ_0-formula $\varphi(x,y)$ such that $I\Sigma_1 \vdash \forall x\ \exists y\ \varphi(x,y)$. Let $B(n) = \exists m > n\ \exists z\ \ (A(a,z,m) \wedge \forall y < z\ \neg\varphi(m,y))$. We have just shown that for every nonstandard $n \in M$, $M \vDash \neg B(n)$. Hence there is $n^* \in \mathbb{N}$ such that $M \models \neg B(n^*)$ (otherwise it would contradict the induction instance for the Σ_1-formula $B(n)$) which means that for all natural numbers $m > n^*$, $\mathbb{N} \models F_a(m) > \min y\varphi(m, y)$. □

A nice feature of all proofs of this kind is that they do not use Gödel's Theorem: we constructed a model $I \models I\Sigma_1 + \neg\,\mathrm{MPG}$ by hands.

Note that in order to establish $I\Sigma_1$-unprovability of our statement we only needed the *ground model* M to satisfy $I\Delta_0 + \exp$ (i.e., $I\Delta_0$ plus the axiom that exponentiation is a total function). However, in order to show that the function is fast-growing we needed M to satisfy the full theory $I\Sigma_1$.

Notice also that MPG easily implies $\mathrm{KM}^{(2)}$ (as well as $\mathrm{PH}^{(2)}$ and 1-consistency of $I\Sigma_1$).

A usual non-model-theoretic argument showing that a certain function F dominates all primitive recursive functions would be as follows. For every $n, k \in \mathbb{N}$, let $A_1(n) = n + 1$, $A_{k+1}(n) = A_k^{(n)}(n)$. Consider the Ackermann function $A(n) = A_n(n)$. It is well-known that $A(n)$ dominates all primitive recursive functions. If a lower bound is found for F in terms of compositions of the Ackermann function with unbounded primitive recursive functions then F dominates all primitive recursive functions. Such is the elementary combinatorial proof by M.Kojman and S.Shelah [**21**] of the fact that the function associated with $\mathrm{KM}^{(2)}$ dominates all primitive recursive functions (and, hence, $\mathrm{KM}^{(2)}$ is $I\Sigma_1$-unprovable).

3. Functions enumerating finite families of sets

Here, we present a statement independent of Peano Arithmetic such that the model-theoretic proof of its PA-unprovability is very simple and appeals directly to the ramseyan reason of unprovability (truth values of formulas as colours).

Let $P(k)$ denote the set of all subsets of $\{0, 1, \ldots, k-1\}$. Let $X \Rightarrow (n)^{k+1}$ be a shorthand for the formula expressing: if for every $x_0 < x_1 < \cdots < x_k$ in X we fix a function $f_{x_0 x_1 \ldots x_k} \colon n \to P(x_0)$ then there is $H \subset X$ of size n such that for all $x_0 < x_1 < \cdots < x_k$ and $x_0 < y_1 < \cdots < y_k$ in H, $f_{x_0 x_1 \ldots x_k} \equiv f_{x_0 y_1 \ldots y_k}$.

Theorem 2. For every $k \in \mathbb{N}$, $I\Sigma_k \nvdash \forall na\, \exists b\, [a, b] \Rightarrow (n)^{k+1}$.

In particular, PA does not prove $\forall man\, \exists b\, [a, b] \Rightarrow (n)^m$.

Proof.
Let $M \models I\Sigma_1$, $n, a, b \in M \setminus \mathbb{N}$, $n < a < b$ and $b = \min y[a, y] \Rightarrow (n)^{k+1}$. Let $\psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_n(x_0, x_1, \ldots, x_k)$ be the first n Δ_0-formulas in $(k+1)$ free variables. For every $x_0 < x_1 < \cdots < x_k$ in $[a, b]$, let $f_{x_0 x_1 \ldots x_k} \colon n \to P(x_0)$ be defined as $f_{x_0 x_1 \ldots x_k}(i) = \{p < x_0 \mid M \models \psi_i(p, x_1, \ldots, x_k)\}$. Now, since $[a, b] \Rightarrow (n)^{k+1}$, there is $H = \{c_0, c_1, \ldots, c_{n-1}\}$ such that for all $i_0 < i_1 < \cdots < i_k$ and $i_0 < j_1 < \cdots < j_k$, we have $f_{c_{i_0} c_{i_1} \ldots c_{i_k}} \equiv f_{c_{i_0} c_{j_1} \ldots c_{j_k}}$, i.e., for all $i < n$ and all $p < c_{i_0}$,

$$M \models \psi_i(p, c_{i_1}, \ldots, c_{i_k}) \leftrightarrow \psi_i(p, c_{j_1}, \ldots, c_{j_k}). \qquad (*)$$

Define $I = \sup\{c_i \mid i \in \mathbb{N}\}$. We can show that I is closed under addition and multiplication exactly as we did in the previous section. Let us show that $I \models I\Sigma_k$. Consider a formula $\exists x_1 \forall x_2 \ldots \psi(x, p, x_1, x_2, \ldots, x_k)$ such that $I \models \exists x_1 \forall x_2 \ldots \psi(e, p, x_1, x_2, \ldots, x_k)$ for $\langle e, p\rangle < c_{i_1} \in I$. Again, it is easy to see that the following holds:

$$I \models \exists x_1 \forall x_2 \ldots \varphi(e, p, x_1, \ldots, x_k) \text{ if and only if}$$

$$M \vDash \exists x_1 < c_{i_1+1} \forall x_2 < c_{i_1+2} \dots \varphi(e, p, x_1, \dots, x_k). \qquad (**)$$

Let $x^* \in M$ be such that $M \vDash \begin{pmatrix} x^* \text{ is the minimal } x < c_{i_1} \text{ such that} \\ \exists x_1 < c_{i_1+1} \forall x_2 < c_{i_1+2} \dots \varphi(x, p, x_1, \dots, x_k) \end{pmatrix}$. Then, by $(**)$, $I \vDash x^*$ is the minimal x such that $\exists x_1 \forall x_2 \dots \varphi(x, p, x_1, \dots, x_k)$. Hence $I \vDash I\Sigma_k$. However, as $M \vDash [b = \min y [a, y] \Rightarrow (n)^{k+1}]$ and the relation $x \Rightarrow (z)^w$ is Δ_0,

$$I \vDash I\Sigma_k + \forall y [a, y] \not\Rightarrow (n)^{k+1}.$$

Thus $I\Sigma_k \nvdash \forall an\ \exists b\ ([a, b] \Rightarrow (n)^{k+1})$. □

It is easily provable that $\forall na\ \exists b\ [a, b] \Rightarrow (n)^k$ implies $\mathrm{KM}^{(k)}$. For that, notice that every regressive function satisfies the condition $f(x_0, x_1, \dots, x_{k-1}) < 2^{k \cdot x_0}$. Of course, a simple argument from [**33**] also goes through to show straight away that $\forall na\ \exists b\ [a, b] \Rightarrow (n)^{k+1}$ implies $\mathrm{RFN}_{\Sigma_1}(I\Sigma_k)$. We conjecture that $\forall na\ \exists b\ [a, b] \Rightarrow (n)^{k+1}$ is equivalent to $\mathrm{RFN}_{\Sigma_1}(I\Sigma_k)$.

We have to mention that the 2^{nx}-trick used in this and the previous section appeared in the article [**16**] as well as (implicitly) in the original article [**33**] and in other writings on the subject. We incorporate it into our combinatorial principles and suggest that this becomes a short easy way to teach model-theoretic unprovability proofs.

4. The story of $\mathrm{PH}^{(k)}_{\log^{(n)}}$ and $\mathrm{KM}^{(k)}_{\log^{(n)}}$

Let $\log^{(n)}(x) = \underbrace{\log(\log \dots \log}_{n \text{ times}}(x))$, Let $2_n(x)$ be defined as $2_0(x) = x$, $2_{n+1}(x) = 2^{2_n(x)}$. Also, define $\log^*(m)$ as the minimal n such that $2_n(2) \geq m$. In the notation for the Ramsey number $R(m, k, c)$, m is the size of a homogeneous set, k is the length of tuples and c is the number of colours.

For every function $F(x)$, define

$$\mathrm{PH}^{(k)}_F \leftrightarrow \forall nc \exists N \left[\begin{array}{c} \text{for every } f\colon [N]^k \to c, \text{there is} \\ \text{a homogeneous } H \subseteq N, |H| \geq n, F(\min H) < |H| \end{array} \right].$$

We say that f is F-regressive if for all $x_0 < x_1 < \dots < x_{k-1}$, we have

$$f(x_0, x_1, \dots, x_{k-1}) < F(x_0).$$

Now define

$$\mathrm{KM}^{(k)}_F \leftrightarrow \forall n \exists N \left[\begin{array}{c} \text{for every } F\text{-regressive } f \text{ defined on } [N]^k \text{ there is} \\ \text{a min-homogeneous subset of } N \text{ of size at least } n \end{array} \right].$$

Also, define $\mathrm{PH}_F \leftrightarrow \forall k\ \mathrm{PH}^{(k)}_F$ and $\mathrm{KM}_F \leftrightarrow \forall k\ \mathrm{KM}^{(k)}_F$. It is easy to see that for every strictly increasing F, $\mathrm{PH}_F \to \mathrm{PH}$ and $\mathrm{KM}_F \to \mathrm{KM}$ thus making these statements PA-unprovable. A.Weiermann proved [**47**] that for every $n \in \mathbb{N}$, $\mathrm{PH}_{\log^{(n)}}$ is PA-unprovable but $\mathrm{PH}_{\log^*}$ is PA-provable.

The following interesting result has been proved recently by Gyesik Lee [**24**]:

(1) if $n < k - 1$ then $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ is $I\Sigma_k$-unprovable;

(2) if $n > k - 1$ then $I\Sigma_1$ proves $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$.

The case $n = k - 1$ is at the moment an open problem. Similar theorems hold for the family $\mathrm{PH}^{(k)}_{\log^{(n)}}$:

(1) if $n < k$ then $I\Sigma_k$ does not prove $\mathrm{PH}^{(k+1)}_{\log^{(n)}}$ (A.Weiermann [**47**]);

(2) if $n > k$ then $I\Sigma_1$ proves $\mathrm{PH}^{(k+1)}_{\log^{(n)}}$ (G.Lee [**24**]).

The case $n = k$ is currently an open problem. However, rather complete solutions in the case $n = k = 1$ have been recently obtained by A.Weiermann and G.Lee [**25**]: if A^{-1} is the inverse of the Ackermann function and $\{F_m\}_{m\in\mathbb{N}}$ is the Grzegorczyk hierarchy of primitive recursive functions then:

(1) $I\Sigma_1 \nvdash \mathrm{PH}^{(2)}_{\frac{\log}{A^{-1}}}$;

(2) for every $m \in \mathbb{N}$, $I\Sigma_1 \vdash \mathrm{PH}^{(2)}_{\frac{\log}{F_m^{-1}}}$;

(3) $I\Sigma_1 \nvdash \mathrm{KM}^{(2)}_f$, where $f(x) = x^{\frac{1}{A^{-1}(x)}}$;

(4) for every $m \in \mathbb{N}$, $I\Sigma_1 \vdash \mathrm{KM}^{(2)}_{f_m}$, where $f_m(x) = x^{\frac{1}{F_m^{-1}(x)}}$.

In particular, $\mathrm{KM}^{(2)}_{\log}$ is provable but $\mathrm{PH}^{(2)}_{\log}$ is unprovable.

The reason for $I\Sigma_1$-provablity of $\mathrm{PH}^{(k+1)}_{\log^{(n)}}$ and $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ for large n comes from the Erdös-Rado theorem [**7**], which implies that an upper bound for the Ramsey number $R(m, k+1, m)$ is $2_n(m)$ for some large enough n depending only on k. Given m and c, let $\ell = \max\{m, c\}$. Consider any coloring $f\colon [0, 2_n(\ell)]^{k+1} \to c$. By Ramsey Theorem, there is $H \subset [0, 2_n(\ell)]$ of size at least ℓ which is f-homogeneous. Also,

$$\log^{(n)}(\min H) < \log^{(n)}(2_n(\ell)) = \ell \leq |H|.$$

Thus $I\Sigma_1 \vdash \mathrm{PH}^{(k+1)}_{\log^{(n)}}$.

A similar argument for $I\Sigma_1$-provability of $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ for large n goes as follows: consider n such that $R(m, k+1, m) < 2_n(m)$ for all m. Let f be a $\log^{(n)}$-regressive function defined on $[0, 2_n(m)]$. Then the image of f is contained in $[0, m]$. Hence there is a homogeneous (thus also min-homogeneous) subset $H \subseteq [0, 2_n(m)]$ of size at least m.

To determine the smallest n which makes these principles $I\Sigma_k$-provable is a nontrivial open problem.

The unprovability results of A.Weiermann and G.Lee are obtained entirely by combinatorial means refering to the fast-growing hierarchy of recursive functions. At the moment no obvious connection can be seen between the direct combinatorial approach and the model-theoretic approach to unprovability proofs.

5. Spreading the indiscernibles

In order to return $\mathrm{KM}^{(k)}_{\log^{(n)}}$ to the realm of unprovability, we add a new condition "for all $x, y \in H$, $x < y \to 2^x < y$" on the min-homogeneous set and make the modified $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$, $I\Sigma_k$-unprovable for all $n \in \mathbb{N}$. Of course, an easy application of the Infinite Ramsey Theorem shows our statements to be provable in second-order arithmetic. For a function F, we define

$$A^k_F \;\leftrightarrow\; \forall a n \exists b \left[\begin{array}{c} \text{for every } F\text{-regressive } f \text{ defined on } [a,b]^k \\ \text{there is a min-homogeneous subset } H \\ \text{of } [a,b] \text{ of size at least } n \text{ such that} \\ \text{for all } x, y \in H\ (x < y \to 2^x < y) \end{array} \right].$$

Theorem 3. For every $k, n \in \mathbb{N}$, $I\Sigma_k \nvdash A^{k+1}_{\log^{(n)}}$.

In the proof, we shall use that for all $x > 2^e$, $\log x < \frac{x}{e}$.

Proof.

For the rest of the proof we fix $k, n \in \mathbb{N}$. Let $M \vDash I\Sigma_1$, $d > e > \mathbb{N}$, $a \geq 2_{n+2}(e)$ and let $\psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_e(x_0, x_1, \ldots, x_k)$ be the first e Δ_0-formulas in $(k+1)$ free variables. Let $b > a$ be minimal such that for any $f\colon [a,b]^{k+1} \to [a,b]$ such that $f(x_0, x_1, \ldots, x_k) < \log^{(n)} x_0$, there is a min-homogeneous set H such that $|H| \geq d$ and for all $x, y \in H$, $x < y$ implies $2^x < y$. Define

$$f(x_0, x_1, \ldots, x_k) = \left\langle \begin{array}{c} \{p < \log^{(n+2)} x_0 \mid \psi_1(p, x_1, \ldots, x_k)\} \\ \vdots \\ \{p < \log^{(n+2)} x_0 \mid \psi_e(p, x_1, \ldots, x_k)\} \end{array} \right\rangle.$$

Notice that, since $\log^{(n+1)} x_0 > 2^e$, we have $f(x_0, x_1, \ldots, x_k) < 2^{e \cdot \log^{(n+2)} x_0} < 2^{e \cdot \frac{1}{e} \cdot \log^{(n+1)} x_0} = \log^{(n)} x_0$. By the choice of b, we obtain a set $H \subset [a,b]$, $H = \{c_i\}_{i=1}^{|H|}$, satisfying the following condition $(*)_{n+2}$: $|H| \geq d$, and for any $c_0 < c_1 < \cdots < c_k$, $c_0 < d_1 < \cdots < d_k$ in H, any $p < \log^{(n+2)} c_0$ and any Δ_0-formula ψ in $(k+1)$ free variables, we have

$$2^{c_0} < c_1 \text{ and}$$

$$M \vDash \psi(p, c_1, \ldots, c_k) \;\leftrightarrow\; \psi(p, d_1, \ldots, d_k).$$

Let us show that for $I = \sup\{c_i \mid i \in \mathbb{N}, c_i \in H\}$, we have $I \vDash I\Sigma_k$.

First, notice that for every $m \in \mathbb{N}$, if $x \in I$ then $2_m(x) \in I$: if $x < c_i$ then $2_m(x) < c_{i+m}$.

Now, let $\varphi(x, x_0, x_1, \ldots, x_k)$ be an arbitrary Δ_0-formula and for $p \in I$ we have $I \vDash \exists x \exists x_1 \forall x_2 \ldots \varphi(x, p, x_1, \ldots, x_k)$, i.e., for some $e \in I$,

$$I \vDash \exists x_1 \forall x_2 \ldots \varphi(e, p, x_1, \ldots, x_k).$$

Since for every m, I is closed under $2_m(x)$, there is $i \in \mathbb{N}$ such that $\langle e, p \rangle < \log^{(n+2)} c_i$. Now, by $(*)_{n+2}$, for every $x < \log^{(n+2)} c_i$, the following condition $(**)$ holds: $I \vDash \exists x_1 \forall x_2 \ldots \varphi(x, p, x_1, \ldots, x_k)$ if and only if $M \vDash \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \ldots \varphi(x, p, x_1, \ldots, x_k)$. Since $M \vDash I\Delta_0$, there is $x^* < c_i$ such that

$$M \vDash x^* \text{ is minimal such that } \exists x_1 < c_{i+1} \forall x_2 < c_{i+2} \ldots \varphi(x^*, p, x_1, \ldots, x_k).$$

Then, by $(**)$,

$$I \vDash x^* \text{ is minimal such that } \exists x_1 \forall x_2 \ldots \varphi(x^*, p, x_1, \ldots, x_k).$$

This completes the proof. □

6. Applying the pigeonhole principle and number theory to indiscernibles

Here, we introduce another condition that would make the modified $\mathrm{KM}^{(k+1)}_{\log^{(n)}}$ $I\Sigma_k$-unprovable for all $n \in \mathbb{N}$, $k \geq 2$.

THEOREM 4. For every $n \in \mathbb{N}$, every $k \geq 2$, the statement

$$\forall a m \exists b \left[\begin{array}{c} \text{for every } \log^{(n)}\text{-regressive function } f \text{ defined} \\ \text{on } [a,b]^{k+1}, \text{ there is an } f\text{-min-homogeneous} \\ \text{subset } H \subset [a,b] \text{ of size at least } m \text{ and} \\ \text{such that } 2^{(\text{second element of } H)} < |H| \end{array} \right].$$

is unprovable in $I\Sigma_k$.

PROOF.
Let $M \models I\Sigma_1$ be nonstandard, $a, d \in M \smallsetminus \mathbb{N}$ and $b \in M$ be minimal such that for every $\log^{(n)}$-regressive function f defined on $[a,b]^{k+1}$, there is a min-homogeneous $H \subset [a,b]$ of size at least d and such that $2^{(\text{second element of } H)} < |H|$.

Let $\mathbb{N} < e < \log^{(n+2)} a$ and $\psi_1(x_0, x_1, \ldots, x_k), \ldots, \psi_e(x_0, x_1, \ldots, x_k)$ be the first e Δ_0-formulas in $(k+1)$ free variables. Define, as in the previous section,

$$f(x_0, x_1, \ldots, x_k) = \left\langle \begin{array}{c} \{p < \log^{(n+2)} x_0 \mid \psi_1(p, x_1, \ldots, x_k)\} \\ \vdots \\ \{p < \log^{(n+2)} x_0 \mid \psi_e(p, x_1, \ldots, x_k)\} \end{array} \right\rangle.$$

Notice that again, since $\log^{(n+1)} x_0 > 2^e$, we have $f(x_0, x_1, \ldots, x_k) < 2^{e \cdot \log^{(n+2)} x_0} < 2^{e \cdot \frac{1}{e} \cdot \log^{(n+1)} x_0} = \log^{(n)} x_0$.

Let $H \subset [a,b]$ be an f-min-homogeneous subset such that

$$2^{(\text{second element of } H)} < |H|$$

and $|H| \geq d$. Let us again write down the indiscernibility condition we have for H: for any $c_0 < c_1 < \cdots < c_k$, $c_0 < d_1 < \cdots < d_k$ in H, any $p < \log^{(n+2)} c_0$ and any Δ_0-formula ψ in $(k+1)$ free variables, we have

$$M \models \psi(p, c_1, \ldots, c_k) \ \leftrightarrow\ \psi(p, d_1, \ldots, d_k).$$

In particular, for any $c_1 < \cdots < c_k$ and $d_1 < \cdots < d_k$ in $H \smallsetminus \{\min H\}$ and any Δ_0-formula φ in k free variables,

$$M \models \varphi(c_1, \ldots, c_k) \ \leftrightarrow\ \varphi(d_1, \ldots, d_k).$$

Let us show that for $I = \sup\{c_i \mid i \in \mathbb{N}, c_i \in H\}$, we have $I \models I\Sigma_k$. Once we show that I is closed under exponentiation, we can demonstarte that $I \models I\Sigma_k$ by the same argument as in the previous section, so we omit this argument here.

Now, let $c < d$ be the first two elements of $H \smallsetminus \{\min H\}$. We are going to demonstarate that $2^c < d$. Then, since $k \geq 2$, by indiscernibility, for any pair $x < y$ in $H \smallsetminus \{\min H\}$, we would have $2^x < y$.

Suppose there is $p < \log^{(n+2)} c$ such that

$$2_{n+3}(p) < d \leq 2_{n+3}(p+1).$$

Then, by indiscernibility, all elements of H which are greater than c belong to

$$(2_{n+3}(p), 2_{n+3}(p+1)].$$

However, there are fewer than $2_{n+3}(p+1) - 2$ elements in this interval, while

$$2_{n+3}(p+1) - 2 < 2_{n+3}(\log^{(n+2)} c) - 2 \leq 2^c - 2 < |H \smallsetminus \{\min H, c\}|$$

and, by the pigeonhole principle, there is no space for all elements of $H \setminus \{\min H, c\}$ there. Hence $2^x < y$ for all $x < y$ in $H \setminus \{\min H\}$, hence I is closed under exponentiation. This completes the proof. $\square$

Note that theorems similar to Theorems 3 and 4 can be proved for $\mathrm{PH}^{(k+1)}_{\log^{(n)}}$, with similar proofs.

It can easily be seen from the proof that we can replace the condition $2^c < |H|$ by $q^c < |H|$ for any rational number $q > 1$.

We can further improve Theorem 4 by fusing the pigeonhole argument with the following straightforward number-theoretic observation. We suggest that a version of this observation may be very relevant in other circumstances when we have to estimate the cardinality of a set of indiscernibles.

Let $\ell = \log^{(n+2)} c$ and p_i be the ith prime. Then for every $i \leq \ell$ and any $c_1, c_2 \in H \setminus \{\min H, c\}$, $c_1 \equiv c_2 (\mathrm{mod}\ p_i)$. Hence, the number of elements of H among any K consecutive elements of $[a, b]$ does not exceed

$$A = K \cdot \prod_{i \leq \ell} \frac{K + p_i - 1}{K p_i}.$$

By openning up the brackets in A, we obtain

$$A \leq K \cdot \prod_{i \leq \ell} \frac{1}{p_i} + 1 \leq \frac{K}{\ell!}.$$

Hence the condition we impose on the min-homogeneous set can be weakened to

$$\frac{q^c}{(\log^{(n+2)} c)!} < |H|,$$

where c is the second element of H. Of course, for large n, it is only a slight improvement. If we use the inequality $m \cdot \ln m < p_m$ from [**41**] instead of $m < p_m$ above then we can show that the condition can be weakened further to

$$q^c \cdot \left(\frac{1}{\ln \ell}\right)^{\ell} \cdot \frac{e^{\mathrm{Li}(\ell)}}{\ell!} < |H|,$$

where $\ell = \log^{(n+2)} c$ and c is the second element of H. For that notice that for all $k \in \mathbb{N}$, all $i \leq \ell + k$ and any $c_1, c_2 \in H \setminus \{\min H, c\}$, $c_1 \equiv c_2 (\mathrm{mod}\ p_i)$. Now, $\prod_{i \leq \ell + k} \frac{1}{\ln i} = \exp(-\sum_{i \leq \ell + k} \ln \ln i)$ and

$$\sum_{i \leq \ell + k} \ln \ln i \geq \int_2^{\ell + k'} \ln \ln x dx \quad \text{for some } k' \leq k.$$

But we also have

$$\int_2^{\ell + k'} \ln \ln x dx = (\ell + k') \cdot \ln \ln(\ell + k') - 2 \ln \ln 2 - \int_2^{\ell + k'} \frac{dx}{\ln x} \;\geq\; \ell \cdot \ln \ln \ell - \mathrm{Li}(\ell).$$

The last inequality follows from the growth rate of $\int_2^x \frac{dt}{\ln t}$ and the fact that ℓ is nonstandard (for any $C \in \mathbb{N}$, the increment of $x \ln \ln x$ on $[\ell, \ell + k']$ is greater than the increment of $\int_2^x \frac{dt}{\ln t}$ by more than C).

Other slight improvements can be obtained by applying refined versions of the number-theoretic observation above. We conjecture that the condition $\min H < |H|$ should be enough to guarantee $I\Sigma_k$-unprovability of our principle for all $n, k \in \mathbb{N}$.

7. Future

Time has come to be able to do the following two things which are within reach.

(1) Interbreed the model theory and combinatorics that lead to the independence phenomenon with the hardcore number theory. This should result in formulating many *consistent* number-theoretic statements implying PH or $\mathrm{Con}_{\mathrm{PA}}$ or, hopefully, even stronger statements. This is not as easy as it sounds, since, as we mentioned in the introduction, all of the existing number theory is (to our knowledge) formalizable in Peano Arithmetic.

(2) Of utmost importance for foundations of mathematics would be to find first-order statements in the language of arithmetic independent of strong theories (higher-order arithmetics, type theory, set theories). For that, we shall have to learn to build models of these theories. Here, we would like to mention the pioneering successful attempts by H.Friedman [**9**] [**11**].

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