RESPLENDENT MODELS AND Σ^1_1-DEFINABILITY WITH AN ORACLE

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ABSTRACT. In this article we find some sufficient and some necessary Σ^1_1-conditions with oracles for a model to be resplendent or chronically resplendent. The main tool of our proofs is internal arguments, that is analogues of classical theorems and model-theoretic constructions conducted inside a model of first-order Peano Arithmetic: arithmetised back-and-forth constructions and versions of the arithmetised completeness theorem, namely constructions of recursively saturated and resplendent models from the point of view of a model of arithmetic. These internal arguments are used in conjunction with Pabion's theorem that ensures that certain oracles are coded in a sufficiently saturated model of arithmetic. Examples of applications are provided for the theories DLO (of dense linear orders) and DIS (of discrete linear orders). These results are then generalised to other ω-categorical theories and theories with a unique countable recursively saturated model.

This paper belongs to the study of first-order theories and structures, the old classical subject that was known as 'logic' in the first half of the 20th century, before the emergence and unravelling of the grand research programmes in major areas of logic. We deal with very general questions about first-order structures and their expansions and try to characterise the intrinsically complex notions of resplendent model and chronically resplendent model (whose definitions appeal to such elusive entities as "the set of all statements in any expanded language that are consistent with the elementary diagram of the model") in an intuitive and simple way. Good examples of the shape of results we aim for are in Theorem 13 for ω_1-saturated dense linear orders: if $(A, <)$ is chronically resplendent and S is a completion of PA then $(A, <)$ is isomorphic to the rational numbers of some model $M \vDash S$, or in Theorem 10 for dense linear orders: if $(A, <)$ is the rational numbers of some model $M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th} \mathbb{N}$ then $(A, <)$ is resplendent. Another simple example (for arbitrary recursive theories T) follows from Lemma 6 and Proposition 4: let Sat_T be a Δ_2^{PA} formula that in every model of $\mathrm{PA} + \mathrm{Con}_T$ strongly interprets a model of T. Consider the following $\Sigma^1_1(\mathbf{0}'')$-formula Φ: "the $\mathcal{L}$-structure is strongly interpreted by Sat_T in a model $N \vDash \mathrm{PA}$ which is strongly interpreted by $\mathrm{Sat}_{\mathrm{PA}}$ in some model $M \vDash \mathrm{PA} + \Pi_2 \operatorname{Th} \mathbb{N}$". For any $\mathcal{L}$-structure A, if $A \vDash \Phi$ then A is a parameter-free resplendent model of T.

Our final theorems "for certain formulas Φ, if $A \vDash \Phi$ then A is resplendent" or "if A is chronically resplendent then $A \vDash \Phi$" may look purely (abstract) model-theoretic but there is a model of Peano Arithmetic hiding inside each of our formulas Φ. The reason that models of arithmetic appear in the study of resplendent and recursively saturated models dates back to S. Kleene: Kleene's Theorem (every recursive set of formulas has a finite axiomatisation in a larger language) was proved by means of explicitly writing down a statement saying "there exists a model of arithmetic such that ... ".

The final results of the article can also be viewed in the context of determining the complexity of the notion of a resplendent model in a conventional scale that extends the arithmetical and analytical hierarchies by allowing oracles to be present in the formulas. Some of the results are lower bounds on complexity of versions of the notion, some separate different versions by establishing a simple notion between them. However, the more important aspect is that a relationship is established between entirely *external* properties of models such as "a structure A is resplendent" and properties whose description is partly *internal*, such as "A is strongly interpreted in a model $M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th} \mathbb{N}$".

This paper, among other things, illustrates the occasional situation where conducting our model theory inside a model of PA brings extra benefits (when the models that we are studying can be made intertwined with the structure of our universe, a model of PA). The main (totally new) idea in this article is to arithmetise existence of recursively saturated and resplendent models, and use these versions of the arithmetised completeness theorem in conjunction with arithmetised back-and-forth arguments and Pabion's theorem to infer results about models 'in the real world'. The main value of the article is in its methods, and we expect that these methods can be re-applied repeatedly in other contexts in model theory, e.g. in situations where the collection of all countable structures of a certain type is classifiable or somehow describable (ω-categorical structures and theories with a unique countable recursively saturated model that we are dealing with in this article are just a particular case of this more general situation).

This article is one of a number of spin-offs from the author's PhD thesis 'On order-types of models of arithmetic' [2] and was written in 2003 and improved in 2005. It can also be considered as a sequel to the article [4], joint with Richard Kaye. I would like to thank Jim Schmerl for reading and extensively commenting on a draft of this paper and for improving the oracle used in this paper (in Theorems 9, 12 and 16) from the author's original $\mathbf{0}''$ to the best possible $\mathbf{0}'$.

1. PRELIMINARIES

A structure A in a finite language $\mathcal{L}$ is called resplendent if for any formula $\varphi(\overline{x})$ in any larger finite language $\mathcal{L}' \supset \mathcal{L}$ and any finite tuple $\overline{a} \in A$, we have: if for some $B \succ A$, B is expandable to an $\mathcal{L}'$-structure satisfying $\varphi(\overline{a})$ then A is expandable to an $\mathcal{L}'$-structure satisfying $\varphi(\overline{a})$. Equivalently, we can define this notion as follows: A is resplendent if for every Σ^1_1-sentence $\Phi(\overline{a}) = \exists R_1 \ldots R_n\ \varphi(R_1, \ldots, R_n, \overline{a})$, where $\overline{a} \in A$ and $\varphi(R_1, \ldots, R_n, \overline{x})$ is a formula in the language $\mathcal{L} \cup \{R_1, \ldots, R_n\}$, we have: if $\mathrm{Th}(A, \overline{a}) + \Phi(\overline{a})$ is consistent then there are relations $R^A_1, \ldots, R^A_n$ on A such that $(A, R^A_1, \ldots, R^A_n, \overline{a}) \vDash \varphi(R_1, \ldots, R_n, \overline{a})$. A structure A is chronically resplendent if for every Σ^1_1-sentence $\Phi(\overline{a})$ consistent with $\mathrm{Th}(A, \overline{a})$, A is expandable to a *resplendent* model of $\mathrm{Th}(A, \overline{a}) + \Phi(\overline{a})$. Intuitively, resplendent models are rich enough and regular enough to be able to incorporate all possible properties of the form "there exists an automorphism with certain properties" or "there exist extra operations that can be introduced on the structure", etc, whose presence does not contradict the theory of the model. E.g., a dense linear order without end-points which is resplendent can be expanded to an ordered field.

For countable structures, the notions of resplendent, chronically resplendent and recursively saturated model coincide. In general, the question whether every resplendent model is chronically resplendent is on open problem.

In a way, the notion of a resplendent model serves to replace many interesting notions of saturation. However, unlike saturated models, resplendent models exist in abundance in all cardinalities (without assuming any amount of GCH) and, more generally, the model theory of resplendent models doesn't normally require any set theory.

Many important properties of models can be expressed as recursive conjunctions of first-order formulas. Using the following fact from [9], we can express such properties by means of Σ^1_1-formulas.

Theorem (S. Kleene). Let C be a recursively enumerable class of formulas in a finite language $\mathcal{L}$ which is closed under predicate calculus deduction and has only infinite models. Then there can be written (effectively in the code of C) a formula Φ in the language $\mathcal{L} \cup \{P\}$, where P is a predicate symbol not occurring in $\mathcal{L}$, such that for every $\mathcal{L}$-formula φ, $\varphi \in C \iff \Phi \vdash \varphi$.

Perhaps a more understandable presentation of this theorem is in the paper [5] by Craig and Vaught. We shall often use the following version of Kleene's theorem: if $\{\varphi_i(\overline{x})\}_{i\in\omega}$

is a recursive collection of formulas in a finite language $\mathcal{L}$ then there is a Σ^1_1-formula $\Phi(\overline{x})$ such that in all infinite $\mathcal{L}$-structures A, $A \vDash \forall \overline{x} \left[\Phi(\overline{x}) \leftrightarrow \bigwedge_{i\in\omega} \varphi_i(\overline{x})\right]$. We shall use this theorem very often to write Σ^1_1-statements expressing things like $\exists M\ (M \vDash \mathrm{PA} \wedge \cdots)$.

Let $\mathbf{d}$ be a Turing degree. By a $\Sigma^1_1(\mathbf{d})$-sentence we shall mean an expression of the form $\exists R_1 \ldots R_n\ \bigwedge_{i\in\omega} \varphi_i(R_1,\ldots,R_n)$, where $\{\ulcorner\varphi_i(\overline{R})\urcorner \mid i \in \omega\}$ is recursive in $\mathbf{d}$. For $\mathbf{d} \neq \mathbf{0}$, $\Sigma^1_1(\mathbf{d})$-sentences may have additional expressive power. A property P of models is Σ^1_1 (Δ^1_n, $\Sigma^1_1(\mathbf{d})$, etc) if there is a Σ^1_1- (Δ^1_n-, $\Sigma^1_1(\mathbf{d})$-, etc) closed formula Φ such that A possesses P if and only if $A \vDash \Phi$.

An oracle we shall often appeal to is $\Pi_n \operatorname{Th}\mathbb{N} = \{\varphi \in \Pi_n \mid \mathbb{N} \vDash \varphi\}$, which has arithmetical complexity $\overline{0^{(n)}}$ and belongs to the Turing degree $\mathbf{0}^{(n)}$. Define $\operatorname{Th}\mathbb{N}$ as $\{\varphi \in \mathcal{L}_{\mathrm{PA}} \mid \mathbb{N} \vDash \varphi\}$. Superficially, it may seem that the use of notions like $\Pi_1 \operatorname{Th}\mathbb{N}$ or $\operatorname{Th}\mathbb{N}$ would amount to subscribing to some questionable philosophy. However, we don't do that, and think of all philosophical questions light-heartedly, as is customary in the subject, by fixing an external universe (a model of some favourite set theory of the day) and assuming that all definitions and constructions take place from the point of view of that model.

If T is an M-finite set of sentences coded by $t \in M$ then $\operatorname{Con}(T)$ is its consistency statement (with parameter t). Similarly, if T is a recursively axiomatised theory then Con_T or $\operatorname{Con}(T)$ is a statement of its consistency. Although these are slightly different notions, we use Con_T and $\operatorname{Con}(T)$ interchangeably. (The difference between the two notions comes when T has an infinite set of axioms in which case in every nonstandard model, nonstandard instances of axioms of T will play additional role (to prove more things within M), in addition to deductions of nonstandard length. If T is finitely axiomatisable then the two formulas are equivalent.)

For every $n \in \mathbb{N}$, there is a canonically constructed formula $\operatorname{Sat}_{\Sigma_n}(x,y)$ such that for every $M \vDash \mathrm{PA}$ and every formula $\varphi(\overline{z}) \in \Sigma_n$, we have $M \vDash \forall \overline{a}\,[\operatorname{Sat}_{\Sigma_n}(\ulcorner\varphi\urcorner, \langle\overline{a}\rangle) \leftrightarrow \varphi(\overline{a})]$.

We say that an $\mathcal{L}$-structure A is strongly interpreted in $M \vDash \mathrm{PA}$ if there are two formulas $\operatorname{Dom}_A(x)$ and $\operatorname{Sat}_A(x,y)$ such that the domain of A is $\{x \in M \mid M \vDash \operatorname{Dom}_A(x)\}$ and for every formula $\varphi(\overline{x}) \in \mathcal{L}$, $A \vDash \varphi(\overline{a})$ if and only if $M \vDash \operatorname{Sat}_A(\ulcorner\varphi\urcorner, \langle\overline{a}\rangle)$. The arithmetised completeness theorem says that for every definable in M set T of $\mathcal{L}$-sentences such that $M \vDash \operatorname{Con}(T)$, there is a model of T strongly interpreted in M. The theorem is proved by conducting the usual completeness proof inside M. For finitely axiomatised theories T, this theorem belongs to D. Hilbert and P. Bernays [8], the generalisation to recursively axiomatised theories is due to Wang Hao [16]. For a discussion of arithmetised completeness, see C. Smorynski's paper [15], for its early history see S. Feferman's [6].

If A is strongly interpreted in $M \vDash \mathrm{PA}$ then for the following M-definable sets

$$\operatorname{Eldiag}^M(A) = \{\varphi(\overline{a}) \in M \mid M \vDash \operatorname{Sat}_A(\varphi, \langle\overline{a}\rangle)\},\quad \operatorname{Th}^M A = \{\varphi \in M \mid M \vDash \operatorname{Sat}_A(\varphi, \langle\rangle)\},$$

we have $M \vDash \operatorname{Con}(\operatorname{Eldiag}^M(A))$ and $M \vDash \operatorname{Con}(\operatorname{Th}^M(A))$.

If a model $M \vDash \mathrm{PA}$ is strongly interpreted in $N \vDash \mathrm{PA}$ then there is a canonical embedding of N onto an initial segment of M.

We use the words "a structure A in a language $\mathcal{L}_1$ is interpreted in a structure B in a language $\mathcal{L}_2$" in their usual meaning: there are $\mathcal{L}_2$ formulas that define analogues of $\mathcal{L}_1$ relations so that the structure B with new relations is isomorphic to A.

Throughout the article, the assumption $M \vDash \mathrm{PA}$ can often be weakened to $M \vDash I\Sigma_1$ or M is a model of some other theory but since there is no principal difference, we shall stay with PA.

The letters DLO stand for $\operatorname{Th}(\mathbb{Q},<)$, the theory of dense linear orders without end-points, DIS stands for $\operatorname{Th}(\mathbb{N},<)$, the theory of discrete linear orders with a left-hand end-point and without a right-hand end-point. For any $A \vDash \mathrm{DLO}$ and any $S \subseteq A$, $\mathrm{DLO}(S)$ is the complete theory of A in the language $\{<\} \cup S$.

A theorem by J.-F. Pabion and D. Richard [10] says: for $M \vDash \mathrm{PA}$, M is ω_1-saturated if and only if the reduct $(M,<)$ is ω_1-saturated. The same statement is true in all uncountable cardinalities κ and is J.-F. Pabion's Theorem [10]: for $M \models \mathrm{PA}$, M is κ-saturated if and only if $(M, <)$ is κ-saturated.

Let us now list what is known so far about complexity of the class of resplendent models. S. Smith proved in [14] that resplendency is Δ^1_2 and that recursive saturation is Σ^1_1. His paper also reports the result of A. Lachlan that resplendency is not Σ^1_1.

In [2], the following fact was proved by the author:

Fact 1. If $(A, <)$ is a dense linear order without end-points interpreted in $M \models \mathrm{PA}$ then A is (definably in M) isomorphic to $Q(M)$, the linearly ordered set of all nonzero fractions of M. Moreover, if $k \in M$ and $a_1 < \cdots < a_k$ are elements of A then $(A, <, a_1, \ldots, a_k)$ is (definably in M) isomorphic to $(Q(M), <, 1, \ldots, k)$.

This is proved by conducting inside M a usual back-and-forth construction (or a one-way forth-construction) of an isomorphism between any countable $A \models \mathrm{DLO}$ and $(\mathbb{Q}, <)$. It is easy to see that in this setup, the notions of 'interpreted' and 'strongly interpreted' coincide since quantifier-elimination for DLO can be conducted in Peano Arithmetic (and in much weaker theories).

In [4], a class of theories was introduced such that every model of PA interprets a unique model of such theory. For such theories T, the following $\Sigma^1_1(\mathbf{0}')$-statement "there is a model M of $\mathrm{PA} + \Pi_1 \,\mathrm{Th}\,\mathbb{N}$ such that A is isomorphic to the unique model of T [strongly] interpreted in M" implies that A is resplendent. The current paper, among other things, generalises this result to arbitrary theories and proves versions of it for parameter-free resplendent models and for several variations of chronically resplendent models.

Also proved in [4] is the fact that resplendency for DIS is not $\Sigma^1_1(C)$ for any oracle $C \subseteq \mathbb{N}$. Moreover, for any theory T in a language containing "$<$" such that every model of T contains an initial segment isomorphic to $\mathbb{N}$, resplendency is not $\Sigma^1_1(C)$ for any oracle $C \subseteq \mathbb{N}$.

2. Main Lemmas

For the rest of the article we fix a finite language $\mathcal{L}$.

Definition

Let $M \models \mathrm{PA}$. We say that $C \subseteq M$ is a Σ^M_1-subset of M (or "is Σ^M_1-definable") if there is $\varphi \in M$ such that $M \models$ "$\varphi(x)$ is a parameter-free Σ_1 formula in the one free variable shown" and $C = \{z \in M \mid M \models \mathrm{Sat}_{\Sigma_1}(\varphi, z)\}$.

Definition

Let $M \models \mathrm{PA}$. We say that an $\mathcal{L}$-structure A is M-recursively saturated if

1. A is strongly interpreted in M;
2. for any collection $p(x, \overline{y}) = \{\varphi_i(x, \overline{y}) \mid i \in M, M \models \text{“}\varphi_i \in \mathcal{L}\text{”}\}$ that is Σ^M_1-definable in M and any $\overline{a} \in A$ such that
$$M \models \forall m \exists x \in A(\bigwedge_{i=1}^{m} \mathrm{Sat}_A(\varphi_i, \langle x, \overline{a}\rangle)),$$
there is $x \in A$ realising $p(x, \overline{a})$, i.e., such that $M \models \forall i\, \mathrm{Sat}_A(\varphi_i, \langle x, \overline{a}\rangle)$.

We use "Σ^M_1" instead of "Δ^M_1" in order to recognise Σ^M_1-formulas as syntactical objects (not to worry about existence of a Π_1 and a Σ_1 formula in M and establishing their M-equivalence). By Craig's Theorem both approaches lead to equivalent classes of formulas.

Example: every $\mathbb{N}$-recursively saturated model is recursively saturated.

Now notice that once you are given formulas $\mathrm{Sat}_A(x, y)$ and $\mathrm{Dom}_A(x)$ strongly interpreting A in M, the property "A is M-recursively saturated" can be expressed by a single

$\mathcal{L}_{\mathrm{PA}}$-sentence:

$$\mathrm{RS}_A \leftrightarrow \forall\varphi\,\forall a\,[\varphi\in\Sigma_1\ \wedge\varphi \text{ defines a set of } \mathcal{L}\text{-formulas } \{\varphi_i(w,v_1,\dots,v_k)\}_{i\in M}\wedge$$
$$\wedge\,\forall m\,\exists x\,\mathrm{Sat}_A\left(\ulcorner\bigwedge_{i=0}^{m}\varphi_i\urcorner,\langle x,(a)_1,\dots,(a)_k\rangle\right)\ \to$$
$$\to\ \exists x^*\,\forall m\,\mathrm{Sat}_A(\ulcorner\varphi_m\urcorner,\langle x^*,(a)_1,\dots,(a)_k\rangle)].$$

Definition

Let $M\vDash\mathrm{PA}$. We say that an $\mathcal{L}$-structure A is M-resplendent if

(1) A is strongly interpreted in M;
(2) for any M-finite language $\mathcal{L}'\supset\mathcal{L}$, any $\varphi\in M$ and any coded tuple $\overline{a}\in A$ of the same arity as φ such that $M\vDash$ "$\varphi(\overline{x})\in\mathcal{L}'$" $\wedge$ "$\overline{a}\in A$" $\wedge$ $\mathrm{Con}(\mathrm{Th}^M(A,\overline{a})+\varphi(\overline{a}))$, there is an expansion of A to a structure A^* in $\mathcal{L}'$ strongly interpreted in M such that $A^*\vDash\varphi(\overline{a})$.

Simple examples.

(1) $Q(M)$ is an M-resplendent model of DLO. Indeed, if $\overline{a}\in Q(M)$ and a formula Φ in the language $\{<,\overline{a},\overline{R}\}$ is consistent with $\mathrm{DLO}(\overline{a})$, conduct the arithmetised completeness process inside M to build a model $A\vDash\mathrm{DLO}(\overline{a})+\Phi$. The $<$-reduct of A is isomorphic to $Q(M)$ by Fact 1, and hence $Q(M)$ is already expandable to a model of Φ.
(2) $M+Q(M)(M^*+M)$ is the unique M-recursively saturated model of DIS (this can be done by imitating inside M the proof that $\mathbb{N}+\mathbb{Q}\mathbb{Z}$ is recursively saturated[1]). Of course there are other models of DIS interpreted in M but an M-recursively saturated model turns out to be unique.

   It was showed by the author in [2] that the order-type of every model of PA strongly interpreted in another model $M\vDash\mathrm{PA}$ is $M+Q(M)(M^*+M)$. The current paper generalises this result and puts it into a wider context.

The following lemma is a generalisation of the Arithmetised Completeness Theorem.

Lemma 2.

(1) If T is a recursively axiomatised $\mathcal{L}$-theory then there is a parameter-free Δ_2^{PA}-formula which in every model $M\vDash\mathrm{PA}+\mathrm{Con}_T$ strongly interprets an M-recursively saturated model of T.
(2) Let $M\vDash\mathrm{PA}$, T be an $\mathcal{L}$-theory coded by $t\in M$ and $M\vDash\mathrm{Con}(T)$. Then there is a Δ_2^M-formula with parameter t which strongly interprets an M-recursively saturated model of T.
(3) If T is a recursively axiomatised $\mathcal{L}$-theory then there is a parameter-free Δ_2^{PA}-formula which in every model $M\vDash\mathrm{PA}+\mathrm{Con}_T$ strongly interprets an M-resplendent model of T.
(4) Let $M\vDash\mathrm{PA}$, T be an $\mathcal{L}$-theory coded by $t\in M$ and $M\vDash\mathrm{Con}(T)$. Then there is a Δ_2^M-formula with parameter t which strongly interprets an M-resplendent model of T.
(5) Let T be a recursively axiomatised $\mathcal{L}$-theory. Then there there is a parameter-free Δ_2^{PA}-formula which in every model $M\vDash\mathrm{PA}+\mathrm{Con}_T$ strongly interprets an M-resplendent model $A\vDash T$ in the language $\mathcal{L}$ and some relations in the language $\{R_1,R_2,\dots\}$ such that for all i, if φ is the ith formula and $M\vDash\mathrm{Con}(\mathrm{Th}_{\mathcal{L}}A+\varphi(R_i))$ then $\varphi(R_i)\in\mathrm{Th}\,A$.

[1]Here and elsewhere in the paper, the product of linear orderings is written lexicographically, that is $(a_1,b_1)<(a_2,b_2)$ if $a_1<a_2$ or $a_1=a_2$ and $b_1<b_2$.

Proof. Let us sketch a proof of (1). The proof repeats the proof of the arithmetised completeness theorem with an additional step of realising all recursive types.

Fix the domain of the future model, say $A = \{x \mid x = x\}$. Let us fix a recursive enumeration of all (Gödel numbers of) $(\mathcal{L} \cup A)$-formulas as $\psi_1(\overline{a_1}), \ldots, \psi_i(\overline{a_i}), \ldots$ and recursively enumerate all (Gödel numbers of) Turing machines as $p_1, \ldots, p_i, \ldots$.

Put $T_0 = T$. Suppose for $i = 2k$, a theory T_i has been defined. Consider $\psi_k(\overline{a_k})$ and put $T_{i+1} = T_i + \psi_k(\overline{a_k})$ if $\mathrm{Con}(T_i + \psi_k(\overline{a_k}))$ and $T_{i+1} = T_i + \neg\psi_k(\overline{a_k})$ otherwise.

Now consider the number p_k. If

(1) the program p_k generates a sequence $\varphi_1(x, \overline{a}), \ldots, \varphi_j(x, \overline{a}), \ldots$ of $\mathcal{L}$-formulas with one free variable x and a finite tuple $\overline{a} \in A$ of constants;
(2) for all j, we have $\mathrm{Con}(T_{i+1} + \exists x \bigwedge_{\ell<j} \varphi_\ell(x, \overline{a}))$

then put $T_{i+2} = T_{i+1} + \{\varphi_1(b, \overline{a}), \ldots, \varphi_j(b, \overline{a}), \ldots\}$, (where b is the first constant symbol from A that hasn't been used yet) and $T_{i+2} = T_{i+1}$ otherwise.

Put $\mathrm{Sat}(\varphi, a) \leftrightarrow \exists i\ \varphi(a) \in T_i$. Clearly $\neg\mathrm{Sat}(\varphi, a) \leftrightarrow \exists i \neg\varphi(a) \in T_i$, and hence Sat is a Δ_2^{PA}-formula.

Parts (2), (3) and (4) are proved similarly, arithmetising the usual constructions.

Part (5). Build an M-resplendent model normally at even steps, but at each odd step consider a new formula $\varphi(R_i)$ (that is the ith formula φ is considered at step $(2i + 1)$ with the relation symbol R_i in it). If $\varphi(R_i)$ is consistent with the previously chosen formulas, from the point of view of M, then add $\varphi(R_i)$ into $\mathrm{Th}\, A$, otherwise do nothing.

Now, in M, if $\varphi(R)$ is consistent with $\mathrm{Th}_{\mathcal{L}}\, A$ then $\varphi(R)$ is consistent with $\mathrm{Th}\, A$ by interpolation, so when $\varphi(R_i)$ appeared in the enumeration at its corresponding turn, it was consistent with previously chosen formulas and hence included into $\mathrm{Th}\, A$. □

In particular, Lemma 2 implies that for every theory T which has an arithmetical axiomatisation (i.e., if $T <_T 0^{(n)}$ for some n), there are recursively saturated models of T strongly interpreted in $\mathbb{N}$.

However, parts (3) and (4) of Lemma 2 are not really necessary since the two notions (being M-resplendent and being M-recursively saturated) coincide. Although not used much in the rest of the paper (apart from the two examples defining a formula in Propositions 7 and 8 below), this fact is worth knowing.

Lemma 3. Let $M \models \mathrm{PA}$. An $\mathcal{L}$-structure A is M-recursively saturated if and only if A is M-resplendent.

Proof. For every Σ_1^M-subset $C \subset M$ such that $M \models$ "$C \subset \mathcal{L}_T$", there can be written, effectively, using only the code of C, a formula Φ_C in the language $\mathcal{L} \cup \{P\}$ such that $M \models \forall\varphi \in \mathcal{L}\ (C \vdash_{pc} \varphi \leftrightarrow \Phi_C \vdash_{pc} \varphi)$, where $\vdash_{pc}$ means deducibility in the predicate calculus. This is an immediate reformulation of Kleene's Theorem and is proved in exactly the same way.

Now, if A is M-resplendent, $\overline{a} \in A$, and $p(x, \overline{a})$ is a Σ_1^M-definable set of $\mathcal{L}$-formulas and $M \models \mathrm{Con}(\mathrm{Eldiag}(A) + \bigwedge_{i=1}^{\infty} p(x, \overline{a}))$ then $M \models \mathrm{Con}(\mathrm{Th}(A, a) + \exists x \Phi_p(x, \overline{a}))$, and hence, since A is M-resplendent, $A \models \exists x \Phi_p(x, \overline{a})$. Thus, A is M-recursively saturated.

Let A be M-recursively saturated, Φ_0 be a sentence in a larger M-finite language $\mathcal{L}'$ possibly containing an M-finite tuple of elements of A and such that $M \models \mathrm{Con}(\mathrm{Th}^M(A, \overline{a}) + \Phi_0)$. Let $\{\Psi_i(x) \mid i \in M\}$ be an enumeration of all $\mathcal{L}'(A)$-formulas with one free variable. Suppose we have already chosen $\Phi_1, \ldots, \Phi_m \in \mathcal{L}'(A)$ such that the following condition (*) holds:

$$M \models \forall\varphi\ a\ \left[\text{``}\varphi \in \mathcal{L}\text{''} \wedge \forall i < \mathrm{len}a\ (a)_i \in A \wedge \mathrm{Pr}_{pc}\left(\bigwedge_{i=0}^{m} \Phi_i \to \varphi(\overline{a})\right) \ \to\ \mathrm{Sat}_A(\varphi, a)\right].$$

Let us show that Φ_{m+1} can be chosen as $\forall x\ \neg\Psi_{m+1}(x)$ or $\Psi_{m+1}(a)$ for some $a \in A$ so that condition (*) still holds.

Let $\overline{c} = c_1, \ldots, c_k$ ($k \in M$) be all the constants from A appearing in $\bigwedge_{i=0}^m \Phi_i$, $\overline{d}$ be the tuple of new constants in Ψ_{m+1}. Consider the following M-enumerable set of formulas

$$p(x) = \{\bigwedge_{i=1}^k x \neq c_i\} \cup \left\{\theta(x, \overline{c}, \overline{d}) \mid M \vDash \mathrm{Pr}_{pc}\left(\bigwedge_{i=0}^m \Phi_i + \Psi_{m+1}(x) \to \theta(x, \overline{c}, \overline{d})\right)\right\}.$$

By the same syntactic manipulation as in the proof of the Ressayre-Barwise-Schlipf Theorem ([12], [1]), we can show that if neither $\bigwedge_{i=0}^m \Phi_i + \forall x \neg \Psi_m(x)$ nor any $\bigwedge_{i=0}^m \Phi_i + \Psi(c_j)$ ($j = 1, 2, \ldots, k$) satisfies condition (*) then $p(x)$ is a type.

By an M-version of Craig's Theorem and M-recursive saturation of A, there is $a^* \in A$ realising $p(x)$. Now, notice that ($M \vDash$) for every $\mathcal{L}$-formula $\varphi(x, \overline{c}, \overline{d})$, if $\bigwedge_{i=0}^m \Phi_i + \Psi_m(x, \overline{c}, \overline{d}) \to \varphi(x, \overline{c}, \overline{d})$ then $\varphi(x, \overline{c}, \overline{d}) \in p(x)$, and hence $A \vDash \varphi(a^*, \overline{c}, \overline{d})$, which means exactly that $\bigwedge_{i=0}^m \Phi_i + \Psi_{m+1}(a^*)$ satisfies condition (*).

Define an $\mathcal{L}'$-structure strongly interpreted in M as follows: the domain is A, an $\mathcal{L}'(A)$ formula Φ is true if and only if there is $m \in M$ such that $\Phi = \Phi_m$. □

3. From M-resplendent models to parameter-free resplendent models

Let us first see what M-resplendency implies for a structure 'in the real world'.

Proposition 4. Let $n \geq 1$, $M \vDash \mathrm{PA} + \Pi_n \mathrm{Th}\,\mathbb{N}$. If A is an M-resplendent $\mathcal{L}$-structure which is (parameter-free) strongly interpreted in M by means of a standard Δ_n^{PA}-formula then A is parameter-free resplendent.

Proof.
Consider a closed formula $\varphi \in \mathcal{L}' \supseteq \mathcal{L}$ such that $\mathbb{N} \vDash \mathrm{Con}(\mathrm{Th}\,A + \varphi)$.

Clearly $M \vDash \Sigma_{n+1} \mathrm{Th}\,\mathbb{N}$. (If $\mathbb{N} \vDash \exists x \psi(x)$, where $\psi \in \Pi_n$, consider $n \in \mathbb{N}$ such that $\mathbb{N} \vDash \psi(n)$. Now, $M \vDash \psi(n)$ because $M \vDash \Pi_n \mathrm{Th}\,\mathbb{N}$.)

Notice that for $\mathrm{Th}^M A = \{x \in M \mid M \vDash \mathrm{Sat}(x, \langle\,\rangle)\}$, we have

$$\mathrm{Th}^M A \cap \mathbb{N} = \mathrm{Th}\,A$$

since $M \vDash \Sigma_{n+1} \mathrm{Th}\,\mathbb{N}$.

Since $\mathbb{N} \vDash \mathrm{Con}(\{x \mid \mathrm{Sat}(x, \langle\,\rangle)\} + \varphi)$, we have $M \vDash \mathrm{Con}(\{x \in M \mid \mathrm{Sat}(x, \langle\,\rangle)\} + \varphi)$, that is $M \vDash \mathrm{Con}(\mathrm{Th}^M A + \varphi)$. Now, because A is M-resplendent, A is expandable to a model of φ. □

Actually, in this proposition, we only needed $\mathrm{Th}^M A$ to be Δ_n^{PA}, not the whole relation "$A \vDash \varphi(\overline{a})$".

First, let us present some formulas with the oracle $\mathbf{0}''$ that imply that A is parameter-free resplendent, using the fact that M-resplendent models are strongly interpreted by means of a Δ_2^{PA} formula.

Corollary 5. The property "an $\mathcal{L}$-structure A is parameter-free resplendent" is implied by a $\Sigma_1^1(\mathbf{0}'')$-formula: "there is a model $K \vDash \mathrm{PA} + \Pi_2 \mathrm{Th}\,\mathbb{N}$ such that A is isomorphic to the structure defined by $\mathrm{Sat}_{\mathcal{L}}$, $\mathrm{Dom}_{\mathcal{L}}$ in K", where $\mathrm{Sat}_{\mathcal{L}}$ and $\mathrm{Dom}_{\mathcal{L}}$ are the standard Δ_2^{PA} formulas that define an M-resplendent model of the empty $\mathcal{L}$-theory in every model of PA. Also, for any recursive $\mathcal{L}$-theory T, the property "a model $A \vDash T$ is parameter-free resplendent" is implied by the same $\Sigma_1^1(\mathbf{0}'')$-formula with $\mathrm{Sat}_{\mathcal{L}}$ and $\mathrm{Dom}_{\mathcal{L}}$ replaced by Sat_T and Dom_T, the Δ_2^{PA} formulas that strongly interpret a model of T in every model of $\mathrm{PA} + \mathrm{Con}_T$.

Another approach to obtaining such a formula would be through the arithmetised version of the classical fact "every structure which is strongly interpreted in a nonstandard model of PA is recursively saturated".

Lemma 6. Suppose $M \vDash \mathrm{PA}$, $N \vDash \mathrm{PA}$ is strongly interpreted in M and an $\mathcal{L}$-structure A is strongly interpreted in N. Then A is M-recursively saturated.

Proof.
Suppose A is strongly interpreted in N by Sat_A and N is strongly interpreted in M by Sat_N. Now, A is strongly interpreted in M by a formula defined as $\mathrm{Sat}(\varphi, a) \leftrightarrow \mathrm{Sat}_N(\ulcorner \mathrm{Sat}_A^N(x,y)\urcorner, \langle \varphi, a\rangle)$.

Given $p(x,\overline{y})$, a Σ_1^M-set of $\mathcal{L}^M$-formulas with free variables $x,\overline{y}$, let us take $\overline{a} \in A$ and suppose that $p(x,\overline{a})$ is a type, i.e., for every $m \in M$,

$$N \vDash \mathrm{Sat}_A(\ulcorner \exists x \bigwedge_{\varphi\in p, \varphi<m} \varphi(x,\overline{y})\urcorner, \langle\overline{a}\rangle).$$

Take a Δ_1^M-set q of $\mathcal{L}^M$-formulas which is logically equivalent to p. Let us show that q is coded in N (by conducting a very general M-version of a proof of "every computable subset of $\mathbb{N}$ is coded in every nonstandard model of PA"). Suppose

$$q = \{x \in M \mid M \vDash \exists y\, Q(x,y)\} = \{x \in M \mid M \vDash \forall z\, S(x,z)\},$$

where Q and S are two M-formulas in the language of arithmetic all of whose quantifiers are bounded. Recall that M is (isomorphic to) an initial segment of N. First, notice that for every $k \in N \setminus M$,

$$\{x \in M \mid N \vDash \forall z < k\ S(x,z)\} \subseteq q \subseteq \{x \in M \mid N \vDash \exists y < k\ Q(x,y)\}.$$

Also notice that for every given $m \in M$,

$$N \vDash \forall x < m\ (\exists y < m\, Q(x,y) \to \forall z < m\ S(x,z)).$$

Therefore, by overspill, there is $m^* \in N \setminus M$ such that for every $x \in M$,

$$N \vDash \exists y < m^*\ Q(x,y)\ \to\ \forall z < m^*\ S(x,z),$$

so the N-coded set

$$\{x \in M \mid \exists y < m^*\ Q(x,y)\} = \{x \in M \mid \exists z < m^*\ S(x,z)\}$$

coincides with q. Now, the set

$$\{m \in N \mid \exists x \in A \text{ such that } A \vDash \bigwedge_{\varphi\in q, \varphi<m} \varphi(x,\overline{a})\}$$

contains M, and hence contains an element $s^* \in N \setminus M$, so there is $x^* \in A$ such that in N:

$$A \vDash \bigwedge_{\varphi\in q, \varphi<s^*} \varphi(x^*,\overline{a}),$$

i.e., x^* realises our type q. □

The ideas of Lemma 6 can be pushed much further: it is possible to build, say, automorphisms of uncountable structures strongly interpreted in a model of arithmetic by repeating (arithmetising) all kinds of results about countable models of various theories, e.g. results of R. Kossak, J. Schmerl, H. Kotlarski, R. Kaye and others on automorphisms of countable recursively saturated structures. The induction that in the original results was of length ω will be of length M, where M is an uncountable model of arithmetic. This story may produce important examples or counterexamples in the study of uncountable models.

Proposition 7. Let an $\mathcal{L}$-theory T be recursively axiomatised. Let $\mathrm{Sat}_{\mathrm{PA}}$ be a standard formula that in every model $M \vDash \mathrm{PA} + \mathrm{Con}_{\mathrm{PA}}$ strongly interprets a model of PA, and Sat_T be a formula that in any model of $\mathrm{PA} + \mathrm{Con}_T$ strongly interprets a model of T. Then the following $\Sigma_1^1(\mathbf{0}'')$-formula implies "a model $A \vDash T$ is parameter-free resplendent": "$\exists f\ \exists M \vDash \mathrm{PA} + \Pi_2\,\mathrm{Th}\,\mathbb{N}$ such that in the model N strongly interpreted in M by means of $\mathrm{Sat}_{\mathrm{PA}}$, the $\mathcal{L}$-structure strongly interpreted in N by means of Sat_T is isomorphic to A by means of f".

Proposition 8. The following $\Sigma^1_1(\mathbf{0}'')$-formula implies "A is parameter-free resplendent": "$\exists f\ \exists M \models \mathrm{PA}+\Pi_2\,\mathrm{Th}\,\mathbb{N}$ such that f is an isomorphism between A and the model defined by Sat and $M \models \mathrm{RS}_A$", where Sat is a Δ_2 formula that strongly interprets an $\mathcal{L}$-structure in every model of PA and RS_A is the statement of M-recursive saturation of A defined earlier.

Let us now improve the oracle to $\mathbf{0}'$ using Lemma 2 (5).

Proposition 9. The following $\Sigma^1_1(\mathbf{0}')$-formula implies "A is parameter-free resplendent": "A is isomorphic to the $\mathcal{L}$-structure that is strongly interpreted in some model $M \models \mathrm{PA} + \Pi_1\,\mathrm{Th}\,\mathbb{N}$ by means of the formula from Lemma 2 (5)".

Proof. Suppose that $\varphi(R_i)$ is consistent with $\mathrm{Th}_{\mathcal{L}}\,A$. Then it is consistent with the (finite) set of formulas in $\mathrm{Th}\,A$ that occurred in the construction of $\mathrm{Th}\,A$ before step $(2i+1)$. Now, since $M \models \Pi_1\,\mathrm{Th}\,\mathbb{N}$, $M \models$ "$\varphi(R_i)$ is consistent with the set of formulas in $\mathrm{Th}\,A$ that were constructed before $\varphi(R_i)$". Then, by Lemma 2 (5), $\varphi(R_i) \in \mathrm{Th}\,A$ (in the expanded language), so A has already been expanded to a model of $\varphi(R_i)$ during the arithmetised construction of A inside M. □

We have learnt that if the Δ_2 formula that strongly interprets an M-resplendent model A in M already contains all information about all possible consistent statements in expanded languages then the oracle $\mathbf{0}'$ is enough to ensure parameter-free resplendency of A. But if the formula is just the normal formula that is only responsible for defining the elementary $\mathcal{L}$-diagram of A in M then we need a stronger oracle $\mathbf{0}''$, to check consistency statements.

4. Dense linear orders

Dense linear orders that are ω_1-saturated are an important class of linearly ordered sets. Every model of nonstandard analysis has a dense ω_1-saturated order-type. Moreover, every parameter-free resplendent model of DLO which is ω_1-saturated is expandable to a model of nonstandard analysis, and even more. Here, we shall see how ω_1-saturation affects our study of resplendent models.

To begin the story of dense linear orders, let us first reproduce a theorem from [4]:

Theorem 10.
If $(A,<) \cong Q(M)$ for some $M \models \mathrm{PA} + \Pi_1\,\mathrm{Th}\,\mathbb{N}$ then $(A,<)$ is resplendent.

Proof.
Consider $\overline{a} \in A$ and a formula $\varphi(\overline{x}) \in \mathcal{L}' \supseteq \mathcal{L}$ such that $\mathbb{N} \models \mathrm{Con}(\mathrm{Th}(A,\overline{a}) + \varphi(\overline{a}))$, that is $\mathbb{N} \models \mathrm{Con}(\mathrm{DLO}(\overline{a}) + \varphi(\overline{a}))$. Since $M \models \Pi_1\,\mathrm{Th}\,\mathbb{N}$, we have $M \models \mathrm{Con}(\mathrm{DLO}(\overline{a}) + \varphi(\overline{a}))$, that is $M \models \mathrm{Con}(\mathrm{Th}^M(A,\overline{a}) + \varphi(\overline{a}))$.

As in Example 1 above, there is $B \models \mathrm{DLO}(\overline{a}) + \varphi(\overline{a})$ strongly interpreted in M. Let $\overline{b} \in B$ be the interpretation of the constants $\overline{a}$. By Fact 1, $(B,<,\overline{b}) \cong (A,<,\overline{a})$, and hence $(A,<,\overline{a})$ is expandable to a model of $\varphi(\overline{a})$. □

In a moment (in Theorem 12) we shall see that if we use a construction from Lemma 2 (5) instead of plain arithmetised completeness then we can guarantee that $(A,<)$ possesses a parameter-free version of chronic resplendency which looks much stronger than resplendency. There are three parameter-free versions of the notion: to allow the first expansion only to parameter-free formulas, only the second one or both.

Definition

(1) A structure A is chronically resplendent I if for every parameter-free sentence $\Phi(\overline{R})$ consistent with $\mathrm{Th}(A)$, A is expandable to a resplendent model of $\mathrm{Th}(A) + \Phi(\overline{R})$.

(2) A structure A is chronically resplendent II if for every sentence $\Phi(\overline{R}, \overline{a})$ consistent with $\mathrm{Th}(A, \overline{a})$, A is expandable to a parameter-free resplendent model of $\mathrm{Th}(A, \overline{a}) + \Phi(\overline{R}, \overline{a})$.
(3) A structure A is chronically resplendent III if for every parameter-free sentence $\Phi(\overline{R})$ consistent with $\mathrm{Th}(A)$, A is expandable to a parameter-free resplendent model of $\Phi(\overline{R})$.

Theorem 11. Let $(A, <) \vDash \mathrm{DLO}$. Then

(1) $(A, <)$ is resplendent if and only if it is parameter-free resplendent;
(2) $(A, <)$ is chronically resplendent if and only if it is chronically resplendent I;
(3) $(A, <)$ is chronically resplendent II if and only if it is chronically resplendent III.

Proof. Let us only show (2), the other two parts are identical. If $A \vDash \mathrm{DLO}$ is chronically resplendent I, $\overline{a} \in A$ and $\varphi(\overline{x}) \in \mathcal{L}' \supseteq \mathcal{L}$ is such that $\mathrm{DLO}(\overline{a}) + \varphi(\overline{a})$ is consistent then the parameter-free statement $\exists \overline{x}(\mathrm{DLO}(\overline{x}) + \varphi(\overline{x}))$ is consistent. Let $(A, <, \overline{R})$ be an expansion of $(A, <)$ to a resplendent model of $\exists \overline{x}(\mathrm{DLO}(\overline{x}) + \varphi(\overline{x}))$, where $\overline{R}$ are all $\mathcal{L}' \setminus \mathcal{L}$-symbols from φ. Let $\overline{b} \in A$ be such that

$$(A, <, \overline{R}) \vDash \mathrm{DLO}(\overline{b}) + \varphi(\overline{b}).$$

Since $(A, <)$ is resplendent, there is an order-preserving automorphism of A that takes $\overline{a}$ to $\overline{b}$. Therefore $(A, <, \overline{a})$ is expandable to a resplendent model of $\varphi(\overline{a})$. □

Now notice that if $(A, <) \cong Q(M)$ for some $M \vDash \mathrm{PA} + \Pi_2 \,\mathrm{Th}\,\mathbb{N}$ then A is chronically resplendent II. Indeed, suppose $(A, <) \cong Q(M)$ for some $M \vDash \mathrm{PA} + \Pi_2 \,\mathrm{Th}\,\mathbb{N}$, $\overline{a} \in A$. Let $\varphi(\overline{R}, <, \overline{a}) + \mathrm{DLO}(\overline{a})$ be consistent. Then $M \vDash \mathrm{Con}(\varphi(\overline{R}, <, \overline{a}) + \mathrm{DLO}(\overline{a}))$. By Lemma 2, there is a standard Δ_2^{PA}-formula defining an M-resplendent model

$$(B, \overline{R}, <, \overline{a}) \vDash \varphi(\overline{R}, <, \overline{a}) + \mathrm{DLO}(\overline{a}).$$

By Fact 1, $(B, <, \overline{a}) \cong (A, <, \overline{a})$. By Proposition 4, $(B, \overline{R}, <, \overline{a})$ is parameter-free resplendent. Therefore $(A, <)$ is chronically resplendent II.

However, we can now improve the oracle to $\Pi_1 \,\mathrm{Th}\,\mathbb{N}$, as was suggested to the author by J. Schmerl.

Theorem 12. If $(A, <) \cong Q(M)$ for some $M \vDash \mathrm{PA} + \Pi_1 \,\mathrm{Th}\,\mathbb{N}$ then A is chronically resplendent II.

Proof. Consider a formula $\varphi(R)$ consistent with DLO and apply Lemma 2 (5) to the theory $\mathrm{DLO} + \varphi(R)$, thus building a model $B \vDash \mathrm{DLO} + \varphi(R)$ strongly interpreted in M in a certain way. Now, if the ith formula $\psi(R, S_i)$ is consistent with the previously chosen (finitely-many) formulas of $\mathrm{Th}\,B$ then, since $M \vDash \Pi_1 \,\mathrm{Th}\,\mathbb{N}$, we have $M \vDash$ "$\psi(R, S_i)$ is consistent with the previously chosen formulas of $\mathrm{Th}\,B$", and hence $\psi(R, S_i) \in \mathrm{Th}\,B$, that is B has already been expanded to a model of $\psi(R, S_i)$ during its construction. By Fact 1, $(B, <) \cong (A, <)$ and hence $(A, <)$ is chronically resplendent III. By Theorem 11 above, $(A, <)$ is chronically resplendent II. □

Now, if we restrict ourselves to ω_1-saturated dense linear orders, here is the picture we obtain:

Theorem 13. For ω_1-saturated dense linear orders, chronic resplendency I implies chronic resplendency II. Moreover, the following are listed in the order of implication, that is $(1) \to (2) \to (3) \to (4) \to (5)$:

(1) $(A, <) \vDash \mathrm{DLO}$ is chronically resplendent I;
(2) $(A, <) \cong Q(M)$ for some resplendent $M \vDash \mathrm{PA}$;
(3) for every consistent theory $S \supseteq \mathrm{PA}$, $(A, <) \cong Q(M)$ for some $M \vDash S$;
(4) $(A, <) \cong Q(M)$ for some $M \vDash \mathrm{PA} + \Pi_1 \,\mathrm{Th}\,\mathbb{N}$;
(5) $(A, <)$ is chronically resplendent II.

Proof. $(1) \to (2)$ is straightforward, $(3) \to (4)$ follows from consistency of PA $+$ $\Pi_1 \operatorname{Th} \mathbb{N}$, $(4) \to (5)$ is Theorem 12 above.

Let us show $(2) \to (3)$. Suppose $M \vDash$ PA is resplendent, $A \cong Q(M)$. Let us first show that $(M, <)$ is ω_1-saturated. Suppose that in M, we have:

$$a_1 < a_2 < \cdots < a_n < \cdots \quad \cdots < b_n < \cdots < b_2 < b_1, \qquad n \in \omega.$$

As $Q(M)$ is ω_1-saturated, there are $c, d \in M$ such that

$$\frac{a_1}{1} < \frac{a_2}{1} < \cdots < \frac{a_n}{1} < \cdots < \frac{c}{d} < \cdots < \frac{b_n}{1} < \cdots < \frac{b_1}{1}.$$

Introduce the initial segments (cuts) $I = \sup_{i\in\omega} a_i$, $J = \inf_{i\in\omega} b_i$ and put $e = \left[\frac{c}{d}\right]$. Clearly, $e \cdot d \le c < (e+1)\cdot d$. Notice that $e + 1 > I$ because for all $x \in I$, $c > x \cdot d$, thus $e > I$ because I is a cut. Similarly, $e < J$ because for all $x > J$, $c < x \cdot d$. Therefore $I < e < J$. Thus $(M, <)$ is an ω_1-saturated model of DIS. Now, by the Pabion-Richard Theorem [10], M is ω_1-saturated. Therefore $S \in \operatorname{SSy}(M)$. Let $s \in M$ be a code for S. The following Σ^1_1-formula (with parameter s)

"there is a model $K = (\oplus, \otimes, \ll)$ such that $K \vDash S \ \wedge (Q(M), <) \cong (Q(K), <)$"

is consistent with $\operatorname{Th}(M, s)$ because all countable nonstandard models N of PA have the same $Q(N)$ and hence, since M is resplendent, $(A, <)$ is isomorphic to $Q(K)$ for some $K \vDash S$. □

If in the conclusion of Proposition 4 or Proposition 9, we could obtain "resplendent" in place of "parameter-free resplendent", all five notions in Theorem 13 would coincide and constitute characterisations of chronically resplendent ω_1-saturated dense linear orders.

5. Are resplendent models chronically resplendent?

Notice that we stopped short of proving that resplendency coincides with "being chronically resplendent II" for ω_1-saturated dense linear orders. If a Σ^1_1-formula could be written expressing $\mathbb{N} < x < \operatorname{Cl}\varnothing \smallsetminus \mathbb{N}$ (or $\mathbb{N} < x <$(nonstandard Σ_1-definable points)) for ω_1-saturated models then $\exists M \vDash \text{PA} + \Pi_1 \operatorname{Th}\mathbb{N} \ \wedge \ Q(M) \cong A$ could be implied by a Σ^1_1-formula without an oracle, and hence resplendent $\Rightarrow$ chronically resplendent II by Theorem 13.

Theorem 13 exposes a connection between the chronic resplendence problem and a form of Friedman's 14th Problem [7]. Let $(\star)$ denote the following statement: "The classes $\{(Q(M), <) \mid M \vDash \text{PA}\}$ and $\{(Q(M), <) \mid M \vDash \text{PA} + \Pi_1 \operatorname{Th}\mathbb{N}\}$ coincide". If formulated about $(M, <)$, not about $(Q(M), <)$, this statement $(\star)$ becomes a particular case of Friedman's 14th Problem [7]: "do classes of order-types of nonstandard models of $T \supseteq$ PA coincide for all completions T of PA?" The problem remains unsolved and seems very difficult. Some work has been done by the author on this problem and the results can be found in [2] and [3].

Proposition 14.

(1) If $(\star)$ holds then every resplendent dense linear order is chronically resplendent II.
(2) A resplendent ω_1-saturated element of

$$\{(Q(M), <) \mid M \vDash \text{PA}\} \smallsetminus \{(Q(M), <) \mid M \vDash \text{PA} + \Pi_1 \operatorname{Th}\mathbb{N}\}$$

(if one exists) solves the chronic resplendence problem negatively.

Proof. 1. If $A \vDash$ DLO is resplendent then $A \cong Q(M)$ for some $M \vDash$ PA, so, by $(\star)$, $A \cong Q(N)$ for some $N \vDash \text{PA} + \Pi_1 \operatorname{Th}\mathbb{N}$ and hence is chronically resplendent II by Theorem 13. (No saturation requirement is used.)

2. If $A \in \{(Q(M),<) \mid M \vDash \mathrm{PA}\} \setminus \{(Q(M),<) \mid M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th} \mathbb{N}\}$ was ω_1-saturated and chronically resplendent then by Theorem 13 we could conclude that $A \cong Q(M)$ for some $M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th} \mathbb{N}$, which does not hold by our assumption. □

6. Generalisations and examples

A general version of the following lemma was first proved by J.-F. Pabion in [11].

Lemma 15. Let T be any theory in the language $\{<\}$ containing the axioms of linear order, $M \vDash \mathrm{PA}$. Suppose $A \vDash T$ is strongly interpreted in M and is M-infinite. Then A is ω_1-saturated if and only if M is ω_1-saturated.

Proof.
The direction $(\Leftarrow)$ is straightforward. For $(\Rightarrow)$, first notice that if $Q(M)$ is ω_1-saturated then M is ω_1-saturated as is done above in Theorem 13.

Now, if there is a point in A preceded by $(M^*,<)$ or followed by $(M,<)$ or followed by a copy of $Q(M)$ then M is ω_1-saturated by Pabion's Theorem and the above argument.

Otherwise consider the linear order $A/\sim$, where

$$x \sim y \;\Leftrightarrow\; M \vDash \exists\, n \text{ [there are } n \text{ points between } x \text{ and } y]$$

The linear order $A/\sim$ is interpreted in M. Now notice that $A/\sim$ does not have neighbouring points, so $A/\sim$ is an ω_1-saturated dense linear order interpreted in M, hence M is ω_1-saturated. □

Theorem 16. In the class of ω_1-saturated discrete linear orders, the following are listed in order of implication, that is $(1) \to (2) \to (3) \to (4) \to (5)$:

(1) $(B,<) \vDash \mathrm{DIS}$ is chronically resplendent I;
(2) $(B,<) \cong M + Q(M)(M^* + M)$ for some resplendent $M \vDash \mathrm{PA}$;
(3) for every consistent theory $S \supseteq \mathrm{PA}$, $(B,<) \cong M + Q(M)(M^* + M)$ for some $M \vDash S$;
(4) $(B,<) \cong M + Q(M)(M^* + M)$ for some $M \vDash \mathrm{PA} + \Pi_1 \operatorname{Th} \mathbb{N}$;
(5) $(B,<)$ is chronically resplendent III.

Proof. $(1) \to (2)$ and $(3) \to (4)$ are straightforward, $(2) \to (3)$ follows from the proof of Theorem 13 and Lemma 15. Let us prove $(4) \to (5)$.

Suppose $\varphi \in \mathcal{L}' \supseteq \mathcal{L}$ is consistent with $\mathrm{DIS} = \operatorname{Th} B$. Then $M \vDash \mathrm{Con}(\mathrm{DIS} + \varphi)$. Now apply Lemma 2 (5) to find a Δ_2^{PA}-formula $\mathrm{Sat}_{\mathrm{DIS}+\varphi}$ which interprets an M-resplendent model of $\mathrm{DIS} + \varphi$ in every model of $\mathrm{PA} + \mathrm{Con}(\mathrm{DIS} + \varphi)$ and keeps track of all statements in finite languages extending $\mathcal{L}'$ consistent with the elementary diagram of this model of $\mathrm{DIS} + \varphi$. By Proposition 9, the model $A \vDash \mathrm{DIS} + \varphi$ defined in such a way inside M is parameter-free resplendent.

To show that $(A,<) \cong M + Q(M)(M^* + M)$, notice that there is only one M-recursively saturated model of DIS, namely $M + Q(M)(M^* + M)$. Indeed, every element of A comes with its $(M^* + M)$-block around it and once $a, b \in A$ belong to different $(M^* + M)$-blocks, and $a < b$ then the type $p(x) = \{a + m < x < b - m \mid m \in M\}$ is realised. Therefore the set of $(M^* + M)$-blocks is densely ordered and hence is isomorphic to $Q(M)$ by Fact 1. This proves that B is chronically resplendent III. □

Definition

A theory T in the language $\mathcal{L}_T = \{R_1, \ldots, R_n\}$ is called treatable if

(1) T has a unique countable recursively saturated model;
(2) there are $\mathcal{L}_{\mathrm{PA}}$-formulas $\mathrm{Dom}(x)$, $\varphi_{R_1}(\overline{x}), \ldots, \varphi_{R_n}(\overline{x})$ such that for every $M \vDash \mathrm{PA}$, the model $\mathrm{Canon}(M,T)$ defined by formulas $\mathrm{Dom}(x)$, $\varphi_{R_1}(\overline{x}), \ldots, \varphi_{R_n}(\overline{x})$ satisfies T;

(3) for every $M \models \mathrm{PA}$ and any $N, K \models \mathrm{PA}$ strongly interpreted in M, we have $\mathrm{Canon}(N,T) \cong \mathrm{Canon}(K,T)$.

(The last clause is a version of uniqueness of the M-recursively saturated model of T.)

A theory T in $\mathcal{L}_T = \{R_1, \ldots, R_n\}$ is categorically treatable if T has a unique countable model and in every $M \models \mathrm{PA}$, for any two models $(A, R_1^A, \ldots, R_n^A) \models T$ and $(B, R_1^B, \ldots, R_n^B) \models T$ interpreted in M and any $a_1, \ldots, a_m \in A$, $b_1, \ldots, b_m \in B$ ($m \in \mathbb{N}$) of the same $\mathcal{L}_T$-type, we have $(A, R_1^A, \ldots, R_n^A, a_1, \ldots, a_m) \cong (B, R_1^B, \ldots, R_n^B, b_1, \ldots, b_m)$. (In this definition we don't require for $\overline{a}$ and $\overline{b}$ to be of the same $\mathcal{L}_T^M$-type or for the isomorphism to be internal but in all our examples they happen to be.)

Categorically treatable theories are roughly the ω-categorical theories whose ω-categoricity can be somehow established in every model of PA. The role that $Q(M)$ plays for DLO is then played by the M-atomless boolean algebra for the theory of atomless boolean algebras, the M-random graph for the theory of the random graph, etc. Treatable theories are roughly the theories with a unique countable recursively saturated model (and this is true inside every model of arithmetic). E.g. DIS has a unique M-recursively saturated model interpreted in any model M of PA, namely $M + Q(M)(M^* + M)$.

Theorem 17. Theorems 10, 11, 12, 13: $(1) \to (2)$ and $(3) \to (4) \to (5)$ hold for any categorically treatable theory. Theorem 16: $(1) \to (2)$ and $(3) \to (4) \to (5)$ holds for any treatable theory.

The proofs directly follow the proofs of Theorems 10 - 16. Essentially, these are the results from the previous sections whose proofs do not have to invoke Pabion's phenomenon.

Apart from generalisations to treatable and categorically treatable theories, there is a number of other potential versions and generalisations of Theorems 4 - 13. However, since the aim of this article is only to present the *method* of obtaining such theorems, we do not list these suggestions here.

Let us finish the article with a remark based on J. Schmerl's words in a recent exchange with the author that fits well with the rest of this paper. Let us say that a resplendent model is 1-resplendent. We say that M is $(n+1)$-resplendent if for every $\overline{a} \in M$ and any $\varphi(R, \overline{a})$ consistent with $\mathrm{Th}(M, \overline{a})$, there is an expansion of M to an n-resplendent model of $\mathrm{Th}(M, \overline{a}) + \varphi(R, \overline{a})$. It is possible to define ∞-resplendent models by continuing this process through all ordinals (by saying that for limit α, a model is α-resplendent if it is β-resplendent for all $\beta < \alpha$) or by exploiting the notion of a totally resplendent model from [13] (this definition is to be used during arithmetisation). It can be shown that, if $M \models \mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$ is resplendent then it is ∞-resplendent (build an ∞-resplendent model of M strongly interpreted in M and isomorphic to M: isomorphism can be ensured by resplendency and the fact that a recursively saturated model M encodes $\mathrm{Th}(M, \overline{a})$ for every $\overline{a} \in M$ (thus every recursively saturated model of PA strongly interprets, with a parameter, its own copy) while the actual construction of a totally resplendent model would be an enhanced version of Lemma 2). It immediately follows from this remark, using the proof of Theorem 13 that if $(A, <) \models \mathrm{DLO}$ is ω_1-saturated and 3-resplendent then $(A, <)$ is ∞-resplendent. Step 1 is to show that $(A, <)$ is isomorphic to $Q(M)$ for some chronically resplendent model $M \models \mathrm{PA}$. Step 2 is to show that $(A, <)$ is isomorphic to $Q(N)$ for some resplendent model $N \models \mathrm{PA} + \Pi_1 \,\mathrm{Th}\, \mathbb{N}$ and Step 3 is to infer ∞-resplendence of $Q(N)$ from ∞-resplendence of N.

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