New results on unprovability and logical strength

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May 30, 2007

Abstract

This is a preprint for Max-Planck Institute's proceedings.

We present recent results in the area of unprovability. We find unprovable statements about a general class of functions using similtaneous diophantine approximation, unprovability results about the Riemann zeta-function using almost-periodicity and probabilistic arguments and a simple unprovability result about converging series using existence of a universal series, that is a sum of terms of the form $a_i \cdot x^{k_i} \cdot y^{\ell_i}$ whose re-arrangement converges to an arbitrary function of two variables.

This reseach takes its origin from H.Friedman's sine principle [2]. The statement roughly says that the sine-function, when considered discretely with rational steps, returns to smaller and smaller neighbourhoods of the same point arbitrarily often. The main idea behind the unprovability proof of the sine-principle is that it was possible to somehow formulate Ramsey-theoretic statements in this set-up without actually mentioning quantification over all possible colourings. It turns out that it is possible to extend this result to larger classes of functions in place of sine, to results of separate interest. This research also led to a general technology of using universality phenomena to imitate Ramsey-style statements in non-Ramseyan contexts while preserving logical strength and unprovability of the original Ramseyan assertions.

1 Unprovable statements obtained by simultaneous diophantine approximation

Here is a quite general theorem that embraces all relevant ideas. We can think of slightly more general formulations but they wouldn't expose new ideas.

Definition 1. Let f be a function of one real argument, definable in the language of arithmetic. A^n_f is the following statement: "for all m, there is N such that for any sequence $\langle a_i \rangle_{i=1}^N$ of rational numbers, there is $H \subseteq N$ of size m such that for any two m-sequences $i_1 < i_2 < \ldots < i_n$ and $i_1 < k_2 < \ldots < k_n$,

$$|f(a_{i_1} \cdot a_{i_2} \cdot \ldots \cdot a_{i_n}) - f(a_{i_1} \cdot a_{k_2} \cdot \ldots \cdot a_{k_n})| < 2^{-i_1}\text{''}.$$

A_f is the statement $\forall n A^n_f$.

Theorem 1. Let f, g and h be three functions such that for any $N \in \mathbb{N}$ and any small $\varepsilon > 0$,

1. h is a periodic function with period a, continuous on its period;
2. f is such that for any $b_1, b_2, \ldots, b_N$, linearly independent over $a\mathbb{Q}$ and any $c_1, c_2, \ldots, c_N \in [0, a)$, there is $x \in \mathbb{Q}$ such that for all $i \in \{1, 2, \ldots, N\}$,

$$|f(b_i \cdot x) \; mod \; a \; - \; c_i| < \varepsilon;$$

3. g is any continuous function on a subset of $\mathbb{R}$, whose image contains $[d, +\infty)$ for some $d \in \mathbb{R}$.

Then $A_{h \circ f \circ g}$ is unprovable in Peano Arithmetic and $A^{n+1}_{h \circ f \circ g}$ is unprovable in $I\Sigma_n$.

This turns out to be a rich class of functions. During the proof, the reader can think of $h(x)$ as $\sin(x)$ or $\{x\}$, of f as any polynomial and of $g(x)$ as e.g. $\frac{1}{x}$ (but note that g does not have to be a bijection when restricted to any of the subsets of its domain: only continuity and covering some $[d, +\infty)$ is needed).

Proof. For convenience, assume that the range of h is $[0, 1]$. We will show that for any $F\colon [N]^n \to [0, 1]$, there are rational numbers $r_1, r_2, \dots, r_N$ in the range of g such that for all $i_1 < i_2 < \cdots < i_n \leq N$,

$$|F(i_1, \dots, i_n) - h(f(r_{i_1} \cdot r_{i_2} \cdot \dots \cdot r_{i_n}))| < \varepsilon.$$

We shall first find such $a\mathbb{Q}$-linearly independent numbers and then find the required rational numbers close to them. The proof goes by induction. The inductive step is as follows. Suppose we have built $a_1 < a_2 < \ldots < a_m$ as needed. Enumerate all C_m^{n-1}-many sequences $i_1 < i_2 < \ldots < i_{n-1} \leq m$, denote the product of the kth sequence $a_{i_1} \cdot \ldots a_{i_{n-1}}$ by d_k and apply the assumption on f to find a_{m+1} as a linearly independent number in a small neighbourhood of a solution of the simultaneous set of diophantine inequalities:

$$|h(f(d_k \cdot x)) - F(i_1, \dots, i_{n-1}, m + 1)| < \varepsilon.$$

Having constructed $a_1, \dots, a_N$, set our desired $r_1, \dots, r_N$ to be any rationals sitting in a very small neighbourhood of $g^{-1}(a_1), \dots, g^{-1}(a_N)$. □

Actually, a periodic function, continuous on its period can be replaced by any function that takes infinitely (but boundedly-many) values on its period. Now let's turn to some examples.

EXAMPLE 1. f IS A POLYNOMIAL.

Lemma 2. (Hermann Weyl, 1916).
Whenever we have a finite number of polynomials $p_1(x), p_2(x), \dots, p_K(x)$ such that each of them has an irrational coefficient then for any $c_1, c_2, \dots, c_K \in [0, 1)$ and any $\varepsilon > 0$, there exists a sumultaneous natural solution of all inequalities $|\{p_i(x)\} - c_i| < \varepsilon$.

I am also assuming there exists an effective vesrion of Weyl's Theorem. This gives us many new results.

For any nonconstant polynimial $p(x)$ with an irrational coefficient outside $\pi\mathbb{Q}$, the following functions work in our set-up:

$$\sin(p(x_1 \cdot x_2 \cdot \dots \cdot x_n));$$

$$\{p(x_1 \cdot x_2 \cdot \dots \cdot x_n)\};$$

$$\left\{\frac{1}{p(x_1 \cdot x_2 \cdot \dots \cdot x_n)}\right\}.$$

Proof. Enumerate all products as above and define $p_k(x) = p(a_{i_1} \cdot \dots \cdot a_{i_{n-1}} \cdot x)$ for all $k \leq C_m^{n-1}$. Apply the effective version of Weyl's Theorem to this finite set of polynomials to solve our inequalities:

$$|\{p(a_{i_1} \cdot \dots \cdot a_{i_{n-1}} \cdot x)\} - F(i_1, \dots i_{n-1}, m)| < \varepsilon.$$

□

Example 2.

Question 1. What is known about the joint diophantine approximation of fractional parts of other classical functions: $e^x, \ln(x)$, etc ?

I guess we have to ask some diophantine experts by email: e.g. whether there is a nice class of N-sequences $a_1, \dots, a_N$ such that for any sequence $c_1, \dots, c_N \in [0,1)$, the collection of simultaneous inequalities $|\{e^{a_i \cdot x}\} - c_i| < \varepsilon$ has a solution $x \in \mathbb{Q}$.

We need to find more examples of functions f that satisfy condition (2) in Theorem 1.

Example 3. f is an IPR-function of Furstenberg and Weiss

Some words about unprovable statements about p-adic functions HERE.

2 Unprovable statements about converging series

Here are some easy considerations based on Fekete's Theorem. Let us do this for $n = 2$ and $I\Sigma_1$-unprovability, to make life easier in the beginning.

Here is a 2-dimensional version of Fekete's theorem.

Lemma 3. There is a sequence $\langle a_i \rangle_{i \in \omega}$ of rational numbers and two sequences $\langle \ell_i \rangle$ and $\langle k_i \rangle$ of natural numbers (where each ℓ_i and each k_i appears only finitely-many times) such that for any continuous function $f\colon [0,1]^2 \to [1,2]$, there is a sequence $\langle m_k \rangle_{k \in \omega}$ such that

$$S_{m_k}(x,y) = \sum_{i=1}^{m_k} a_i x^{\ell_i} y^{k_i} \longrightarrow_{k \to \infty} f(x,y)$$

uniformly on $[0,1]^2$.

Proof. SKETCH OF THE PROOF HERE. □

Let us introduce two second-order statements P_1 and P_2 and one first-order statement P_3 and show that $\mathrm{RT}^1 \to P_1 \to P_2 \to P_3 \to \mathrm{KM}^2$.

P_1: for any sequence $\langle m_k \rangle_{k \in \omega}$, such that $S_{m_k}(x,y)$ uniformly converges to some function $[0,1]^2 \to [1,2]$, there is an infinite set $H \subseteq \mathbb{N}$ and $k_0 \in \mathbb{N}$ such that for any $i < j < n$ in H and any $k > k_0$, $|S_{m_k}(\frac{1}{i}, \frac{1}{j}) - S_{m_k}(\frac{1}{i}, \frac{1}{n})| < 3^{-i}$.

P_2: for all m there exists N such that for any sequence $\langle m_k \rangle_{k \in \omega}$ such that $S_k(x,y)$ uniformly converges to some function $[0,1]^2 \to [1,2]$, there is a subset $H \subset N$ of size m and k_0 such that for any $i < j < n$ in H and any $k > k_0$, $|S_{m_k}(\frac{1}{i}, \frac{1}{j}) - S_{m_k}(\frac{1}{i}, \frac{1}{n})| < 3^{-i}$.

P_3 is the statement: "for all m there exists N such that for all k there is a subset $H \subseteq N$ such that for all $i < j < n$ in H, $|S_k(\frac{1}{i}, \frac{1}{j}) - S_k(\frac{1}{i}, \frac{1}{n})| < 3^{-i}$".

Proof.

The proof that RT^1 implies $\to P_1$ is easy. Choose any i_1 and put $X_1 = \mathbb{N}$. Suppose $i_1, \dots i_n$ and X_n have been constructed and define i_{n+1} and X_{n+1} as follows. Partition $[1,2]$ into 3^n segments of length 3^{-i} and define I to be a segment that contains infintely-many images of $S_{m_k}(\frac{1}{i_n}, \frac{1}{x})$ $(x \in X_n)$. Put $X_{n+1} = \{x \in X_n \mid S_{m_k}(\frac{1}{i_n}, \frac{1}{x}) \in I\}$ and put $i_{n+1} = \min X_{n+1}$.

Let us show thatP_3 implies KM^2. For any m, consider N as in P_3 and show that this N is suitable for KM^2. Consider a regressive function $F\colon [N]^2 \to N$ and let $f(x,y)$ be defined on $[\{1/N, \dots 1/2, 1\}]^2$ as $f(\frac{1}{i}, \frac{1}{j}) = 1 + \frac{1}{F(i,j)}$ and continue f to a continuous function $f\colon [0,1]^2 \to [1,2]$. Set $\varepsilon = \frac{1}{3N}$. By Lemma 1, choose an index k such that $|S_k(x,y) - f(x,y)| < \varepsilon$ uniformly for all x, y.

Choose $H \subseteq N$ of size m such that for all $i < j < n$ in H, $|S_k(\frac{1}{i}, \frac{1}{j}) - S_k(\frac{1}{i}, \frac{1}{n})| < 3^{-i}$. Then $|f(\frac{1}{i}, \frac{1}{j}) - f(\frac{1}{i}, \frac{1}{n})| \leq |f(\frac{1}{i}, \frac{1}{j}) - S_k(\frac{1}{i}, \frac{1}{j})| + |S_k(\frac{1}{i}, \frac{1}{j}) - S_k(\frac{1}{i}, \frac{1}{n})| + |f(\frac{1}{i}, \frac{1}{n}) - S_k(\frac{1}{i}, \frac{1}{n})| < \varepsilon + 3^{-i} + \varepsilon$.

However, since F is regressive, we know that the distance between $f(\frac{1}{i}, \frac{1}{j})$ and other images of f is at least $\frac{1}{i(i-1)} > 3^{-i} + 2\varepsilon$ for all i. Hence, on H, $f(\frac{1}{i}, \frac{1}{j}) = f(\frac{1}{i}, \frac{1}{n})$ and $F(i,j) = F(i,n)$, so H is the needed min-homogeneous subset for F. $\square$

3 Riemann zeta function

Lemma 4. *Let $\sigma > 1$ and let I_σ be a non-trivial interval (which exists according to [1]) on the real line in which the zeta values $\zeta(\sigma + it)$ are dense. Let $\varepsilon > 0$. Assume that $F : [R]^n \to I_\sigma$. Let N be so large that $|\prod_{i=N+1}^{\infty} \frac{1}{1-p_i^{-\sigma-it}}| < \varepsilon/6$ uniformly in t. Let $K := \mathbb{Q}(\log(p_1), \log(p_2), \dots, \log(p_N))$. Here p_i denotes the i-th prime. Then there exist K-algebraic irrationals $a_1, \dots, a_R$ over K such that $\deg_K(a_i) = 3^{3^{n^i}}$ for $1 \leq i \leq R$ and such that*

$$|F(i_1, \dots, i_n) - \zeta(\sigma + i \cdot a_{i_1} \cdot \ldots \cdot a_{i_n})| < \varepsilon$$

for $1 \leq i_1 < \ldots < i_n \leq R$.

Proof. By [1] find for all sequences $i_1 < \ldots < i_n$ rational numbers $t(i_1, \dots, i_n)$ such that $|\zeta(\sigma + i \cdot t(i_1, \dots, i_n)) - F(i_1, i_2, \dots, q+1)| < \varepsilon/6$.

The proof is now by induction on i_n. Assume $i_n = n$. Pick K-algebraic irrationals $a_1, \dots, a_{n-1}$ such that $\deg(a_i) = 3^{3^{n^i}}$. By [1] we know that $t \mapsto \zeta(\sigma + it)$ is dense in I_σ. Then by density of the algebraic irrationals involved we can pick a suitable a_n which has to satisfy only $|F(1, 2, \dots, n) - \zeta(\sigma + ia_1 \cdot \ldots \cdot a_n)| < \varepsilon$.

Now consider $i_n = q + 1$ and assume that $a_1, \dots, a_q$ have been constructed with the desired properties.

Put $\zeta_N(\sigma + i \cdot t) := \prod_{i=1}^{N} \frac{1}{1 - p_i^{-\sigma - i \cdot t}}$ and $R_N(\sigma + i \cdot t) := \frac{\zeta(\sigma + i \cdot t)}{\zeta_N(\sigma + i \cdot t)}$.

By the condition on degrees the products, $\log(p_j) a_{i_1} \cdot \ldots \cdot a_{i_n}$ are linearly independent. Hence by the effective version of Kronecker's theorem on simultaneous diophantine approximation and continuity considerations, we find a rational number t with effective bounds on nominator and denominator such that

$$|\zeta_N(\sigma + t \cdot a_{i_1} \cdot \ldots \cdot a_{i_{n-1}}) - \zeta_N(\sigma + i \cdot t(i_1, \dots, i_{n-1}, q+1)| < \varepsilon/6$$

for all relevant indices. By continuity, find a positive δ such that for all γ with $|\gamma| \leq \delta$ we have

$$|\zeta(\sigma + i(t + \gamma) \cdot a_{i_1} \cdot \ldots \cdot a_{i_{n-1}})) - F(i_1, i_2, \dots, q+1)| < \varepsilon.$$

Find a_{q+1} of appropriate degree in a δ-neighbourhood of t such that

$$|\zeta(\sigma + i \cdot a_{i_1} \cdot \ldots \cdot a_{i_{n-1}} \cdot a_{i_{q+1}}) - F(i_1, i_2, \dots, q+1)| < \varepsilon/6.$$

We finally obtain

$$\begin{array}{rl} & |\zeta(\sigma+i\cdot a_{i_1}\cdot\ldots\cdot a_{i_n})-F(i_1,\ldots,i_n)| \\ \leq & |\zeta_N(\sigma+i\cdot a_{i_1}\cdot\ldots\cdot a_{i_n})-\zeta_N(\sigma+i\cdot t(i_1,\ldots,i_n))| \\ + & |\zeta(\sigma+i\cdot t(i_1,\ldots,i_n))-F(i_1,\ldots,i_n)| \\ + & |R_N(\sigma+i\cdot a_{i_1}\cdot\ldots\cdot a_{i_n})| \\ + & |R_N(\sigma+i\cdot t(i_1,\ldots,i_n)|<\varepsilon. \end{array}$$

□

Lemma 5. *Let $\sigma>1$ and let I_σ be a non trivial interval (which exists according to [1]) on the real line in which the zeta values $\zeta(\sigma+it)$ are dense. Let $\varepsilon>0$. Assume that $F:[R]^n\to I_\sigma$. Then there exist rationals $r_1,\ldots,r_R$ such that nominators and denominators are effectively controlled for $1\leq i\leq R$ and such that $|F(i_1,\ldots,i_n)-\zeta(\sigma(r_{i_1}\cdot\ldots\cdot r_{i_n})|<\varepsilon$ for $1\leq i_1<\ldots<i_n\leq R$.*

Proof. Apply the previous Lemma and pick the a_i. By Dirichlet's result on simultaneous diophantine approximation we can find nice rationals r_i close to a_i. Then do a continuity argument. □

As before we obtain the following applications.

Theorem 6. *Fix $\sigma>1$ and a natural number $n\geq 2$. For all natural numbers K there exists a natural number R so large that for all rational numbers $r_1<\ldots<r_R$ which nominators and denominators bounded prim rec in R there exists a subset H of cardinality K such that for all choices $r_{i_1}<r_{i_2},\ldots<r_{i_K}\in H$ and all $r_{i_1}<r_{j_2}<\ldots<r_{j_K}\in H$ we have $|\zeta(\sigma+ir_{i_1}\cdot r_{i_2}\cdot\ldots\cdot r_{i_K})-\zeta(\sigma+i\cdot r_{i_1}\cdot r_{j_2}\cdot\ldots\cdot r_{j_K})|<2^{-i_1}$.*

Theorem 7. *Theorem 6 is not provable in* PA.

In the next step we extend the previous theorem to the case where $\sigma\in(\frac{1}{2},1]$. The proof will now have a different flavour since $|\prod_{i=N+1}^{\infty}\frac{1}{1-p_i^{-\sigma-it}}|$ does not converge uniformly for $\sigma\in(0.5,1]$ and any fixed N.

As before, let $R_N(s):=\zeta(s)\cdot\prod_{i=1}^{N}(1-p_i^{-s})$. The following three lemmas can be found in Bohr-Courant [1]

Lemma 8. *Assume that $0.5<\sigma\leq 1$. Assume that ε,δ are strictly positive reals. Then there exists N_0 depending on $\sigma,\varepsilon,\delta$ such that: For every fixed $N>N_0$ we have that for T large enough the sum of the lengths of the intervals $I\subseteq[-T,T]$, such that for $t\in I$ we have $|R_N(\sigma+it)-1|\geq\varepsilon$, is smaller than $\delta\cdot T$.*

We need the following version of Kronecker's Lemma.

Lemma 9. *Assume that $\lambda_1,\ldots,\lambda_N$ are linearly independent over the rationals. Assume that we are given an N-dimensional cube Q,with side length d, contained in the unit cube. Assume that the sides of Q are parallel to the axes. Let $M:=\{t\in[0,T]:(t\lambda_1,\ldots,t\lambda_N)\in Q(\mod 1)\}$. Then M consists of a finite number of intervals. Let $M(T)$ be the sum of the lengths of these intervals. Then $\lim_{T\to\infty}\frac{M(T)}{T}=d^N$.*

Lemma 10. *Assume that $(r_i)_{i=1}^{\infty}$ is a sequence of positive reals such that $\sum_{i=1}^{\infty}r_i$ diverges and such that $\sum_{i=1}^{\infty}r_i$ converges. Let $S_N(\eta_1,\ldots,\eta_N):=\sum_{i=1}^{N}\log(1-r_i\exp(2\cdot\pi\cdot i\cdot\eta_i))$. Assume that I_0 is a finite collection of cubes with non-zero diameter. Then the Lebesgue measure of $\{(\eta_1,\ldots,\eta_N)\in[0,1]^N:S_N(\eta_1,\ldots,\eta_N)\in I_0\}$ is strictly positive.*

Lemma 11. *Let* $0.5 < \sigma \leq 1$. *Let* $\varepsilon > 0$. *Assume that* $F : [R]^n \to [1,2]$. *Let* $N \geq N_0$. *Let* $K := \mathbb{Q}(\log(p_1), \log(p_2), \ldots, \log(p_N))$. *Here* p_i *denotes the* i*-th prime. Then there exist* K*-algebraic irrationals* $a_1, \ldots, a_R$ *over* K *such that* $\deg_K(a_i) = 3^{3^{n^i}}$ *for* $1 \leq i \leq R$ *and such that*

$$|F(i_1, \ldots, i_n) - \zeta(\sigma + i \cdot a_{i_1} \cdot \ldots \cdot a_{i_n})| < \varepsilon$$

for $1 \leq i_1 < \ldots < i_n \leq R$.

Proof. The proof is again by induction on i_n. Assume $i_n = n$. By Bohr-Courant [1] we know that the ζ-values are dense in $\mathbb{C}$ for the vertical lines under consideration. Pick K-algebraic irrationals $a_1, \ldots, a_{n-1}$ such that $\deg(a_i) = 3^{3^{n^i}}$. Then by density of the algebraic irrationals involved we can pick a suitable a_n which has to satisfy only $|F(1, 2, \ldots, n) - \zeta(\sigma + ia_1 \cdot \ldots \cdot a_n)| < \varepsilon$.

Now consider $i_n = q + 1$ and assume that $a_1, \ldots, a_q$ have been constructed with the desired properties.

Put $\zeta_N(\sigma + i \cdot t) := \prod_{i=1}^{N} \frac{1}{1 - p_i^{-\sigma - i \cdot t}}$ and $R_N(\sigma + i \cdot t) := \frac{\zeta(\sigma + i \cdot t)}{\zeta_N(\sigma + i \cdot t)}$. Let $\vec{i} :=:= i_1, \ldots, i_{n-1}$ be a typical sequence of indices $i_1 < \ldots < i_{n-1} \leq n$. By continuity of the involved operations let us fix a positive ε_1 such that for simultaneously for all $\vec{i}$ we have that from $|x - F(\vec{i})| < \varepsilon_1$ and $|y - 1| < \varepsilon_1$ we can conclude that $|x \cdot y - F(\vec{i})| < \varepsilon$. Further we may choose an ε_2 such that we have that for all $\vec{i}$ we can conclude from from $|\log(x) - \log(F(\vec{i})| < \varepsilon_2$ that $|x - F(\vec{i})| < \varepsilon_1$. The assertion follows when we can find an N and a t such that for all $\vec{i}$ we have simultaneously $|\log \prod_{i=1}^{N} (1 - p_n^{-\sigma - i \cdot t \cdot \prod a_{\vec{i}}}) - \log(F(\vec{i}))| < \varepsilon_2$ and $|R_N(\sigma + i \cdot t \cdot \prod a_{\vec{i}}) - 1| < \varepsilon_1$.

Let us introduce some abbreviations. Let $r_n := p_n^{-\sigma}$, $\lambda_{n,\vec{i}} := -\frac{\log p_n \cdot \prod a_{\vec{i}}}{2\pi}$,

$$S_{N,\vec{i}}(\sigma + i \cdot t) := \sum_{n=1}^{N} \log(1 - p_n^{-\sigma - i \cdot t \cdot \prod a_{\vec{i}}}) = \sum_{n=1}^{N} \log(1 - r_n \cdot \exp(2\pi i \cdot t \lambda_{n,\vec{i}})).$$

So we have to fulfill all the inequalities $|S_{N,\vec{i}}(\sigma + i \cdot t) - F(\vec{i})| < \varepsilon_2$ simultaneously. Let us consider the function $S_N(\eta_1, \ldots, \eta_N) := \sum_{n=1}^{N} \log(1 - r_n \cdot \exp(2\pi i \eta_n))$. For every $\vec{i}$ let $Q_{\vec{i}}$ be an N-dimensional cube with center $F(\vec{i})$ and diameter smaller than ε_2. Since $0.5 < \sigma \leq 1$ the sum $\sum_{n=1}^{\infty} r_n$ does diverge and the sum $\sum_{n=1}^{\infty} r_n^2$ does converge. We can thus apply auxiliary lemma 10 to see that the probability that for all $\vec{i}$ the value $S_N(\eta_{1,\vec{i}}, \ldots, \eta_{N,\vec{i}}$ is contained in $R_{\vec{i}}$ is strictly positive, say greater than k. (Note that we can apply here the Fubini theorem to see that the probabilities for different $\vec{i}$ multiply.) The Lemma 10 is here applied to vectors of length N times the number of the $\vec{i}$.

Now we want to apply the Kronecker Lemma in this situation. Let

$$\Omega(\vec{i}) := \{(\eta_1, \ldots, \eta_N) : S_N(\eta_1, \ldots, \eta_N) \in R_{\vec{i}}\}.$$

We have that $S_N(\sigma + i \cdot t \cdot \prod a_{\vec{i}}) \in R_{\vec{i}}$ iff $(t \cdot \lambda_1, \ldots, t \cdot \lambda_n) \in \Omega(\vec{i})$. By choice the numbers $\lambda_{n,\vec{i}} = -\frac{\log(p_n) \cdot \prod a_{\vec{i}}}{2\pi}$ are simultaneously linear independent for all $n = 1, \ldots, N$ and all choices of $\vec{i}$. Let $M(T)$ be the set of those $t \in [0, T]$ such that $(t\lambda_{1,\vec{i}}, \ldots, t\lambda_{N,\vec{i}}) \mod 1 \in \Omega(\vec{i})$ for all $\vec{i}$. Then by Kronecker's Lemma for large enough T the Lebegue measure of $M(T)$ is larger than $k^{\#\vec{i}} \cdot T$.

Let us now consider the remainder terms $R_N(\sigma + i \cdot t \cdot \prod a_{\vec{i}})$.

By applying auxiliary lemma 8 for $\varepsilon = \varepsilon_1$ and $\delta = \frac{k^{\#\vec{i}}}{2 \cdot \#\vec{i}}$ we find a fixed $N > N_0$ such that for each $\vec{i}$ the sum $L_{\vec{i}}$ of the lengths of the intervals consisting of those $t \in [-T, T]$ for which $|R_N(\sigma + i \cdot t \cdot \prod a_{\vec{i}}) - 1| \geq \varepsilon_1$ is less than $\frac{k^{\#\vec{i}}}{2\#\vec{i}}$. Hence the sum (taken over all $\vec{i}$) of the lengths of all intervals consisting of those $t \in [-T, T]$ for which for all $\vec{i}$ simultaneously we have $|R_N(\sigma + i \cdot t \cdot \prod a_{\vec{i}}) - 1| \geq \varepsilon_1$ is less than $\frac{k^{\#\vec{i}}}{2}$. Let $M^*(T)$ denote the set of all such t.

We now choose a $T > T_0$ such that the Lebesgue measure of $M(T)$ is larger than $T \cdot k^{\#\vec{i}}/2$. Now we have that the measure of $M(T)$ is larger then $T\frac{k^{\#\vec{i}}}{2}$ and the measure of $M^*(T)$ is strictly smaller than $\frac{k^{\#\vec{i}}}{2}T$. Therefore there is at least one t in $M(T)$ which is not in $M^*(T)$. For this t we have that for all $\vec{i}$ that $|\zeta(\sigma + i \cdot t \cdot \prod a_{\vec{i}}) - F(\vec{i})| < \varepsilon$. Now by continuity we can choose an an algebraic approximation of t of the right kind. □

Lemma 12. *Let* $0.5 < \sigma \leq 1$ *Let* $\varepsilon > 0$. *Assume that* $F : [R]^n \to [1, 2]$. *Then there exist rationals* $r_1, \ldots, r_R$ *such that nominators and denominators are effectively controlled for* $1 \leq i \leq R$ *and such that* $|F(i_1, \ldots, i_n) - \zeta(\sigma + i \cdot r_{i_1} \cdot \ldots \cdot r_{i_n})| < \varepsilon$ *for* $1 \leq i_1 < \ldots < i_n \leq R$.

Proof. Apply the previous Lemma and pick the a_i. By Dirichlet's result on simultaneous diophantine approximation we can find nice rationals r_i close to a_i. Then do a continuity argument. □

As before we obtain the following applications.

Theorem 13. *Fix* $0.5 < \sigma \leq 1$ *and a natural number* $n \geq 2$. *For all natural numbers* K *there exists a natural number* R *so large that for all rational numbers* $r_1 < \ldots < r_R$ *which nominators and denominators bounded prim rec in* R *there exists a subset* H *of cardinality* K *such that for all choices* $r_{i_1} < r_{i_2}, \ldots < r_{i_n} \in H$ *and all* $r_{i_1} < r_{j_2} < \ldots < r_{j_n} \in H$ *we have* $|\zeta(\sigma + ir_{i_1} \cdot r_{i_2} \cdot \ldots \cdot r_{i_n}) - \zeta(\sigma + i \cdot r_{i_1} \cdot r_{j_2} \cdot \ldots \cdot r_{j_n})| < 2^{-i_1}$.

Theorem 14. *Theorem 13 is not provable in* PA.

Here s a list of further unprovability conjectures. In each of the conjectures, the statement is provable in some stronger theory, that is, known not to be inconsistent.

Conjecture 1. *The following is not provable in* $I\Sigma_1$. *Fix* $\sigma = 0.5$. *For all natural numbers* K *there exists a natural number* R *so large that for all rational numbers* $r_1 < \ldots < r_R$ *which nominators and denominators bounded by a certain primitive recursive function of* R, *there exists a subset* H *of cardinality* K *such that for all choices* $r_{i_1} < r_{i_2} \in H$ *and all* $r_{i_1} < r_{j_2} \in H$ *we have* $|\zeta(\sigma + ir_{i_1} \cdot r_{i_2}) - \zeta(\sigma + i \cdot r_{i_1} \cdot r_{j_2})| < 2^{-i_1}$.

We have a sketch for this assertion, applying a zeta result [4] by Chris Hughes.

Conjecture 2. *The following is not provable in* $I\Sigma_1$. *For all natural numbers* K, *there exists a natural number* R *so large that for all natural numbers* $3 \leq m_1 < \ldots < m_R$, *there exists a subset* H *of cardinality* K *such that for all choices* $m < l < k$ *in* H, *we have* $|\sin(m \cdot l^m) - \sin(m \cdot k^m)| < 2^{-m}$.

Conjecture 3. *The following is not provable in* $I\Sigma_1$. *For all natural numbers* K, *there exists a natural number* R *so large that for all natural numbers* $3 \leq m_1 < \ldots < m_R$, *there exists a subset* H *of cardinality* K *such that for all choices* $m < l < k$ *in* H, *we have* $|\zeta^{(m)}(0.75 + i \cdot l) - \zeta^{(m)}(0.75 + i \cdot k)| < 2^{-m}$. *Here* $\zeta(m)$ *denotes the* m*-th derivative of* ζ.

Conjecture 4. *The following is not provable in* $I\Sigma_1$. *For all natural numbers* K, *there exists a natural number* R *so large that for all natural numbers* $3 \leq m_1 < \ldots < m_R$, *there exists a subset* H *of cardinality* K *such that for all choices* $2 < m < l < k$ *in* H, *we have* $|\zeta(\frac{1}{m} + i \cdot l) - \zeta(\frac{1}{m} + i \cdot k)| < 2^{-m}$.

Conjecture 5. *Let γ_m be an enumeration of the imaginary parts of the zeta's zeros in the upper half-plane in increasing order. The following is not provable in $I\Sigma_1$. For all natural numbers K, there exists a natural number R so large that for all natural numbers $3 \leq m_1 < \ldots < m_R$, there exists a subset H of cardinality K such that for all choices $m < l < k$ in H we have $|\log(\gamma_m)^2 \cdot \log(\gamma_l)^m) - \log(m)^2 \cdot \log(\gamma_k)^m| \mod 1 < 2^{-m}$.*

Conjecture 6. *For every natural number n and every natural number K and every function $F\colon [K] \to [0,1]$, there exist rational numbers $a_1, \ldots, a_K$ and natural numbers $b_1, \ldots, b_K$ such that for all choices of indices $i_1 < \ldots < i_n \leq K$ the following inequality holds: $|\frac{\zeta(\frac{1}{2}+i\cdot a_{i_1}\cdot\ldots\cdot a_{i_n})}{\sqrt{\log(\log(b_{i_n}))}} - F(i_1, \ldots, i_n)| < \frac{1}{2^{-i_1}}$*

Conjecture 7. *The following is not provable in* PA. *For all natural numbers n and all natural numbers K, there exists a natural number R so large that for all rational numbers $r_1 < \ldots < r_R$ whose numerators and denominators are bounded by a certain primitive recursive function of R and for all natural numbers $m_1 < \ldots < m_R$ bounded primive recursively in R, there exists a subset H of cardinality K such that for all choices $r_{i_1} < \ldots < r_{i_n} \in H$ and all $r_{i_1} < r_{j_2} < \cdot \ldots \cdot r_{j_n} \in H$, we have $|\frac{\zeta(\frac{1}{2}+i\cdot r_{i_1}\cdot\ldots\cdot r_{i_n})}{\sqrt{\log(\log(m_{i_n}))}} - \frac{\zeta(\frac{1}{2}+i\cdot r_{i_1}\cdot\ldots\cdot r_{j_n})}{\sqrt{\log(\log(m_{j_n}))}}| < 2^{-i_1}$.*

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