Complex Networks
Problem Sheet 4

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1. Let $G = (V, E)$ be an undirected graph. Initially, each node $v \in V$ is assigned a value $x_v(0)$ in the interval $[0, 1]$. Time is discrete, and nodes update their values synchronously according to the linear recursion

$$x_v(t+1) = \frac{1}{\deg(v)} \sum_{u:(u,v)\in E} x_u(t). \tag{1}$$

   (a) Write down the set of linear equations in (1) in matrix form as $\mathbf{x}(t+1) = P\mathbf{x}(t)$, i.e., specify the elements of the matrix P.

   (b) Compute an invariant distribution corresponding to the stochastic matrix P, i.e., find a solution of $\pi P = \pi$.
   Hint: It turns out the Markov chain with transition probability matrix P is reversible, and you can compute an invariant distribution by solving the local balance equations, which state that

$$\pi_x p_{xy} = \pi_y p_{yx} \quad \forall\, x, y \in V.$$

   If these equations have a probability vector π as a solution, then it is an invariant distribution of the Markov chain.

   (c) Assume that the graph G is connected and non-bipartite. (A graph is bipartite if the vertex set V can be partitioned into disjoint subsets V_1 and V_2, i.e., with $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges between two nodes of V_1 or two nodes of V_2. In other words, $E \subseteq V_1 \times V_2$.) In this case, it is known that the Markov chain with transition probability matrix P is irreducible and aperiodic. Comment on what happens to $\mathbf{x}(t)$ as t tends to infinity.

   (d) What property of a node determines how influential that node is in determining the final outcome of the above process?

2. A graph $G = (V, E)$ is called bipartite if there exist vertex sets X and Y such that $V = X \cup Y$, $X \cap Y = \emptyset$ and $E \subseteq X \times Y$. In words, X and Y partition the vertex set, and there is no edge between two elements of X or two elements of Y.

   (a) Suppose G is a bipartite graph with $|X| = |Y| = n$, and that every vertex has the same degree d. Show that d and $-d$ are both eigenvalues of A_G. (*Hint.* Guess the corresponding eigenvectors of A_G. It may be helpful to write down a small example for yourself, say with $n = 3$ and $d = 2$.)

   (b) Show that A_G violates one of the conclusions of the Perron-Frobenius theorem.

   (c) Which of the conditions of the Perron-Frobenius theorem does the matrix A_G violate? Demonstrate on an example of your choice with at least 4 nodes and $d \geq 2$.

3. Let $X_t, t \geq 0$ be an asymmetric random walk on $\{0, 1, 2, \ldots, n\}$ in continuous time, with transition rates given by $q_{k,k+1} = \lambda$ and $q_{k,k-1} = \mu$ for all $k \in \{1, 2, \ldots, n-1\}$. All other transition rates are zero. In particular, the states 0 and n are absorbing.

   (a) Write down the rate matrix (also known as infinitesimal generator) for this Markov process.

   (b) Show that $M_t = \left(\frac{\mu}{\lambda}\right)^{X_t}$ is a martingale.

   (c) Find the probability that the random walk, started in some state $k \in \{0, 1, 2, \ldots, n\}$, hits state n before state 0.

4. Recall the Wright-Fisher model, which is a discrete time model describing the evolution of a population of N genes. Each gene has two forms, or alleles, which we denote A and a. The population size stays fixed over time. If we let N_t denote the number of A alleles in generation t, then generation $t+1$ is obtained as follows. Each of the N genes in generation $t+1$ is sampled independently, and uniformly at random (with replacement), from the genes in generation t.

   (a) Conditional on $N_t = n$, the number of A alleles in generation $t+1$ has a binomial distribution. What are the parameters of this binomial distribution?

   (b) Use the answer to the last part to show that N_t is a martingale.

   (c) If $N_0 = k$, what is the probability that eventually there are only A alleles in the population? Explain your answer carefully.