1. Let $G = (V, E)$ be an undirected graph. Initially, each node $v \in V$ is assigned a value $x_v(0)$ in the interval $[0, 1]$. Time is discrete, and nodes update their values synchronously according to the linear recursion

$$x_v(t + 1) = \frac{1}{\deg(v)} \sum_{u:(u,v) \in E} x_u(t). \quad (1)$$

(a) Write down the set of linear equations in (1) in matrix form as $x(t + 1) = P x(t)$, i.e., specify the elements of the matrix $P$.

(b) Compute an invariant distribution corresponding to the stochastic matrix $P$, i.e., find a solution of $\pi P = \pi$. 

Hint: It turns out the Markov chain with transition probability matrix $P$ is reversible, and you can compute an invariant distribution by solving the local balance equations.

(c) Assume that the graph $G$ is connected and non-bipartite. (A graph is bipartite if the vertex set $V$ can be partitioned into disjoint subsets $V_1$ and $V_2$, i.e., with $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges between two nodes of $V_1$ or two nodes of $V_2$. In other words, $E \subseteq V_1 \times V_2$.) In this case, it is known that the Markov chain with transition probability matrix $P$ is irreducible and aperiodic. Comment on what happens to $x(t)$ as $t$ tends to infinity.

(d) What property of a node determines how influential that node is in determining the final outcome of the above process?

2. Let $X_t, t \geq 0$ be an asymmetric random walk on $\{0, 1, 2, \ldots, n\}$ in continuous time, with transition rates given by $q_{k,k+1} = \lambda$ and $q_{k,k-1} = \mu$ for all $k \in \{1, 2, \ldots, n-1\}$. All other transition rates are zero. In particular, the states 0 and $n$ are absorbing.

(a) Write down the rate matrix (also known as infinitesimal generator) for this Markov process.

(b) Show that $M_t = \left( \frac{\mu}{\lambda} \right)^X_t$ is a martingale.

(c) Find the probability that the random walk, started in some state $k \in \{0, 1, 2, \ldots, n\}$, hits state $n$ before state 0.
**3.** Consider the following modification of the classical voter model on the complete graph $K_n$. Nodes can be in one of two states, 0 or 1, and change state as follows. Each node $v$ becomes active at the points of a Poisson process of rate $\lambda$, independent of all other nodes. It then contacts a node $w$ chosen uniformly at random from among all $n$ nodes (including itself). If $w$ has the same state as $v$, nothing happens. Otherwise, $v$ copies the state of $w$ with probability $p$, independent of everything in the past; with the residual probability $1 - p$, it retains its current state. (You can think of this as modelling an attachment to one’s current opinion / preference /affiliation.)

Suppose that initially, at time zero, $k$ nodes are in state 1 and $n - k$ nodes are in state 0. Let $T$ denote the random time that the process hits one of the absorbing states, either the all-zero state, denoted 0, or the all-one state, denoted 1.

(a) Compute the probability of hitting the all-one state.

(b) Compute the expectation of $T$, the random time to absorption.

**Hint.** You may, if you wish, use the following facts for the classical voter model; the first was derived in lectures, the second is a known result. These facts are that, for the classical voter model, $\Pr_k(\text{hit } 1) = k/n$, and $\mathbb{E}_k[T] = \frac{1}{\lambda} nh(k/n)$, where, for $x \in [0, 1]$, $h(x) = -x \log x - (1 - x) \log(1 - x)$ denotes the entropy of a Bern($x$) random variable.