Complex Networks
Problem Sheet 5

$**$ **Please hand in solutions to questions 3 and 4 on this sheet.** $**$

1. Consider the following modification of the classical voter model on the complete graph K_n. Nodes can be in one of two states, 0 or 1, and change state as follows. Each node v becomes active at the points of a Poisson process of rate λ, independent of all other nodes. It then contacts a node w chosen uniformly at random from among all n nodes (including itself). If w has the same state as v, nothing happens. Otherwise, v copies the state of w with probability p, independent of everything in the past; with the residual probability $1-p$, it retains its current state. (You can think of this as modelling an attachment to one's current opinion / preference /affiliation.)

   Suppose that initially, at time zero, k nodes are in state 1 and $n-k$ nodes are in state 0. Let T denote the random time that the process hits one of the absorbing states, either the all-zero state, denoted $\mathbf{0}$, or the all-one state, denoted $\mathbf{1}$.

   (a) Compute the probability of hitting the all-one state.

   (b) Compute the expectation of T, the random time to absorption.

   Hint. You may, if you wish, use the following facts for the classical voter model; the first was derived in lectures, the second is a known result. These facts are that, for the classical voter model, $\mathbb{P}_k(\text{hit } \mathbf{1}) = k/n$, and $\mathbb{E}_k[T] = \frac{1}{\lambda} n h(k/n)$, where, for $x \in [0,1]$, $h(x) = -x \log x - (1-x) \log(1-x)$ denotes the entropy of a Bern(x) random variable.

2. Let $G = (V, E)$ be a graph on 4 nodes with the following 5 edges $(1,2)$, $(2,3)$, $(3,4)$, $(4,1)$ and $(1,3)$; in other words, it is a square with one diagonal. Think of each of these edges as two directed edges. Now consider the voter model on this graph where contacts along each directed edge happen according to independent unit rate Poisson processes. Suppose the voter model starts with nodes 1 and 3 in state 1, and nodes 2 and 4 in state 0. Compute the probability that all nodes eventually reach consensus on the value 1.
   Hint. If you need to find an invariant distribution, see if the local balance equations have a solution.

3. Let S_n denote the star graph, which consists of a hub connected to each of $n-1$ leaves; there are no edges between leaves. Consider the following voter model on S_n. Each node becomes active at the points of a Poisson process of rate 1, independent of all other nodes. When it becomes active, it chooses a neighbour uniformly at random from the set of all its neighbours (i.e., excluding itself), and copies the state of that neighbour.

   Denote by $X_v(t) \in \{0,1\}$ the state of node v at time t. Let $M(t) = (n-1)X_{\text{hub}}(t) + \sum_{v \neq \text{hub}} X_v(t)$.

(a) Show that $M(t)$ is a martingale.

(b) Suppose that initially the hub and $k-1$ leaves are in state 1, while $n-k$ leaves are in state 0. What is the probability of being absorbed into the all-1 state?

4. In this problem, we compute an upper bound on the time to reach consensus for the voter model on a star graph, described in Problem 3.

   (a) We would like to describe the voter model, backwards in time, in terms of coalescing random walks. For a single one of these random walks, what are the transition rates (from a leaf to the hub, and from the hub to each leaf)?

   (b) Next, let us consider two of these random walks, started at different leaves, say. We only need to keep track of the distance between the particles performing these random walks. This distance is either 0, 1 or 2, and when it becomes 0, the particles are at the same node and coalesce. Describe the evolution of this distance as a Markov process.

   (c) Compute the expected time for this Markov process to hit state (distance) 0 starting in state 2. This is the expected time for two random walks to meet, and hence for two particles to merge.

   (d) We are interested in the time until each of $n-1$ other particles has merged with a given particle. This is an upper bound on the time to consensus. Using the fact that the expectation of the maximum of non-negative random variables is bounded by the expectation of their sum, obtain an upper bound on the expected time to reach consensus in the voter model on the star.

5. Consider n nodes arranged in a ring, with an edge between each node and its two neighbours. In other words, if the nodes are numbered $0, 1, 2, \ldots, n-1$, node i has edges to nodes $i-1$ and $i+1$ modulo n. Consider the voter model on this graph, where each node becomes active at the points of a unit rate Poisson process (independent of other nodes), chooses one of its two neighbours with equal probability (independent of everything else), and adopts the state of that neighbour. We want to bound the time to consensus in this model.

   (a) Consider two particles, starting at nodes i and j, and performing independent random walks until they meet. Let Y_t denote the clockwise distance from i to j; suppose that initially j lies clockwise of i so that $Y_0 \in \{1, 2, \ldots, n-1\}$. The two particles merge when this distance becomes 0 or n, i.e., at the time $T = \inf\{t > 0 : Y_t = 0 \text{ or n}\}$. Show that $M_t = Y_t^2 - 2t$ is a martingale on the time period $[0, T]$.

   (b) Compute $\mathbb{E}T$, the expected time until the two particles coalesce. You may use the fact that
   $$\mathbb{P}(Y_T = n) = \frac{Y_0}{n}, \quad \mathbb{P}(Y_T = 0) = 1 - \frac{Y_0}{n}.$$

   (c) We are interested in the time until each of $n-1$ other particles has merged with a given particle. This is an upper bound on the time to consensus. Using the fact that the expectation of the maximum of non-negative random variables is bounded by the expectation of their sum, obtain an upper bound on the expected time to reach consensus in the voter model on the cycle graph on n nodes.