Complex Networks

MATH 36201/M6201  Problem Sheet 8  Autumn 2014

** Questions 2 and 3 on this sheet counts towards your final mark for Level M students **

1. Let $K_4$ denote the complete graph on 4 vertices.
   
   (a) Draw $K_4$.
   
   (b) Compute exactly the expected number of copies of $K_4$ in $G(n, p)$, the Erdős-Rényi random graph on $n$ vertices where each edge is present with probability $p$, independent of the others.
   
   (c) Compute the variance of the number of copies of $K_4$ in $G(n, p)$ from first principles. It is enough if your answer gets the correct scaling in $n$ and $p$. You can ignore any constants that don’t depend on $n$ or $p$ in your calculations. You can also ignore terms in $n$ and $p$ that grow more slowly than the dominant term.
   
   (d) Find an $\alpha_c \in (0, \infty)$ (or prove that none exists) such that the following is true:

   $P(G(n, n^{-\alpha})$ contains a copy of $K_4) \rightarrow \begin{cases} 0, & \text{if } \alpha > \alpha_c, \\
   1, & \text{if } \alpha < \alpha_c. \end{cases}$

   Justify your answer fully.

2. Recall that $G = (V, E)$ is bipartite if there exist vertex sets $X$ and $Y$ such that $V = X \cup Y$, $X \cap Y = \emptyset$ and $E \subseteq X \times Y$. In words, $X$ and $Y$ partition the vertex set, and there is no edge between two elements of $X$ or two elements of $Y$.

   The random bipartite graph $G(n, n, p)$ has $2n$ vertices which can be partitioned as $V = X \cup Y$, $X \cap Y = \emptyset$, with $|X| = |Y| = n$. Moreover, each edge in $X \times Y$ is present with probability $p$, independent of the others. There are no edges in $X \times X$ or $Y \times Y$.

   (a) Let $K_{3,3}$ denote the complete bipartite graph on 3+3 vertices. Draw $K_{3,3}$.
   
   (b) Compute exactly the expected number of copies of $K_{3,3}$ in $G(n, n, p)$.
   
   (c) Compute the variance of the number of copies of $K_{3,3}$ in $G(n, n, p)$ from first principles. It is enough if your answer gets the correct scaling in $n$ and $p$. You can ignore any constants that don’t depend on $n$ or $p$ in your calculations. You can also ignore terms in $n$ and $p$ that grow more slowly than the dominant term.
   
   (d) Find an $\alpha_c \in (0, \infty)$ (or prove that none exists) such that the following is true:

   $P(G(n, n, n^{-\alpha})$ contains a copy of $K_{3,3}) \rightarrow \begin{cases} 0, & \text{if } \alpha > \alpha_c, \\
   1, & \text{if } \alpha < \alpha_c. \end{cases}$

   Justify your answer fully.
3. Let $S_k$ denote the star graph on $k$ nodes, consisting of a hub and $k-1$ leaves.

(a) Show that $S_k$ is a balanced graph.

(b) Using the results in your notes for balanced graphs, find a value $\alpha_k \in (0, \infty)$ such that

$$\mathbb{P}(G(n, n^{-\alpha}) \text{ contains a copy of } S_k) \rightarrow \begin{cases} 0, & \text{if } \alpha > \alpha_k, \\ 1, & \text{if } \alpha < \alpha_k. \end{cases}$$

Clearly state the result you will use before using it.

(c) The chromatic number of a graph $G$, denoted $\chi(G)$, is defined as the minimum number of colours required to colour the nodes of the graph in such a way that no two nodes with an edge between them have the same colour. It is known that $\chi(G) \leq d_{\text{max}} + 1$, where $d_{\text{max}}$ denotes the maximum degree of all nodes in $G$. Indeed, a simple greedy algorithm that goes through the nodes in arbitrary order, assigning each node a colour distinct from that of all its already coloured neighbours will achieve a valid colouring using at most $d_{\text{max}} + 1$ colours.

Use this fact to provide either an upper or a lower bound on $\chi(G)$ that holds with high probability for $G$ drawn from $G(n, p)$. You should state your result precisely.