Complex Networks
Solutions to Problem Sheet 2

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1. Let $T_1, T_2, T_3, \ldots$ denote the increment times of the process $\{X_t, t \geq 0\}$. Since $Y_t = X_{ct}$, the increment times of the process Y_t are when ct takes the values $T_1, T_2, T_3, \ldots$, i.e., at times $T_1/c, T_2/c$ and so on.

   Since X_t is a Poisson process with intensity λ, we know that $T_1, T_2 - T_1, T_3 - T_2$, and so on are iid Exp($\lambda$) random variables. Hence, by the answer to Q2(a) in Problem Sheet 1, the random variables T_1/c, $(T_2 - T_1)/c$, $(T_3 - T_2)/c$ are Exp(λc) random variables. That they are mutually independent follows from the fact that functions of independent random variables are independent (see remark below).

   It also remains to check that the process Y_t has independent increments, i.e., that for any $0 < s < t$, $Y_t - Y_s$ is independent of $\{Y_u, u \leq s\}$. But this assertion is the same as the assertion that $X_{ct} - X_{cs}$ is independent of $\{X_u, u \leq cs\}$, which we know to be true because X_t is a Poisson process, and hence has independent increments.

   Remark What does it mean to say that a collection of random variables, $\{X_i, i \in I\}$ are mutually independent?

   Recall what it means for a collection of events $\{A_i, i \in I\}$ to be mutually independent. (The index set I could be finite or infinite.) It means that, for *any* finite subcollection $i_1, i_2, \ldots i_n$ of *distinct* events (you can't pick the same index twice),

   $$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_n}).$$

   Similarly, we say that a collection of random variables are mutually independent if *any* events involving these random variables are mutually independent. As a concrete example, X_1, X_2 and X_3 are mutually independent if the events $X_1 \in B_1$, $X_2 \in B_2$ and $X_3 \in B_3$ for *any* subsets B_1, B_2 and B_3 of the real numbers. So, by asserting independence of these three random variables, we are actually asserting the independence of infinitely many events. It is a very strong assumption.

   Given this definition, it is easy to see that functions of independent random variables are independent. If f_1, f_2 and f_3 are three functions, then

   $$\begin{aligned}\mathbb{P}(f_i(X_i) \in B_i, i = 1, 2, 3) &= \mathbb{P}(X_i \in f_i^{-1}(B_i), i = 1, 2, 3) \\ &= \prod_{i=1}^{3} \mathbb{P}(X_i \in f_i^{-1}(B_i)) = \prod_{i=1}^{3} \mathbb{P}(f_i(X_i) \in B_i).\end{aligned}$$

   Here, $f^{-1}(B)$ denotes the pre-image of B in f, namely, the set $\{x : f(x) \in B\}$. There is no assumption that the function f is invertible, and f^{-1} does not denote the inverse of f.

   Important qualification for those who know measure theory: I said "any subsets" above. This is not quite right. We need to restrict ourselves to subsets which are "measurable", and that in turn depends on what measure space structure we impose on the real numbers. I will not assume knowledge of measure theory in this course. Suffice it to say that just about any subset of $\mathbb{R}$ you are likely to encounter, in this course or most others, will be measurable!

2. As $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ are independent Poisson processes, we have for all $s, t > 0$ that $X_{t+s}^1 - X_s^1$ and $X_{t+s}^2 - X_s^2$ are independent Poisson random variables, with means $\lambda_1 t$ and $\lambda_2 t$ respectively (by one of the definitions of a Poisson process). Hence, by Question 4 of Homework 1, $X_{t+s} - X_s$ is a Poisson random variable with mean $(\lambda_1 + \lambda_2)t$.

   Moreover, the increments of the X^1 process are independent of the past of the X^1 process because it is a Poisson process, and independent of the past of the X^2 process, because these processes are mutually independent. Hence, the increments of the X^1 process are independent of the past of the X process. Likewise for the increments of the X^2 process. Putting these together, the increments of the X process are independent of the past of the X process. This is the other property we need to complete the proof that $X_t, t \geq 0$ is a Poisson process.

3. Denote the successive events in the Poisson processes $X_t^1, t \geq 0$ and $X_t^2, t \geq 0$ by $T_n^1, n \in \mathbb{N}$ and $T_n^2, n \in \mathbb{N}$ respectively, and the events in the superposition $X_t, t \geq 0$ by $T_n, n \in \mathbb{N}$. What can we say about T_1, the time until the first event in the superposition of the two Poisson processes? Clearly, it is the minimum of T_1^1 and T_1^2. Since X^1 and X^2 are Poisson processes, T_1^1 and T_1^2 are exponentially distributed with parameters λ_1 and λ_2 respectively. Moreover, T_1^1 and T_1^2 are independent random variables since the corresponding Poisson processes are independent. Hence, by the answer to HW1, Question 5, T_1 is exponentially distributed with parameter $\lambda = \lambda_1 + \lambda_2$.

   Next, irrespective of whether $T = T_1$ or T_2, the times until the next event of the two Poisson processes are exponentially distributed with parameters λ_1 and λ_2, independent of each other and of the past of the X^1 and X^2 processes (using the memoryless property of the exponential distribution). Hence, by the same reasoning, $T_2 - T_1$ is also an Exp(λ) random variable. The same reasoning applies to subsequent event times in the Poisson process $X_t, t \geq 0$.

   In order to complete the proof that X_t is a Poisson process of rate λ, we need to show that $X_{t+u} - X_t$ is independent of $X_s, s \leq t$ for arbitrary $t, u > 0$. The corresponding property is true of each of the processes X_t^1 and X_t^2 since these are Poisson processes. Moreover $X_{t+u}^2 - X_t^2$ is independent of $X_s^1, s \leq t$, and the same with superscripts 1 and 2 interchanged, since the X^1 and X^2 processes are independent of each other. Summing X^1 and X^2, it follows that the required independence properties hold for the X_t process.

4. First, the moment generating function of each T_i is given by

$$\begin{aligned} M_i(\theta) &:= E\left[e^{\theta T_i}\right] = \int_0^\infty e^{\theta t} \lambda e^{-\lambda t} dt \\ &= \begin{cases} \frac{\lambda}{\lambda - \theta}, & \text{if } \theta < \lambda, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

   Hence, we obtain the conditional expectation

$$E\left[e^{\theta T} \;\middle|\; N = n\right] = E\left[e^{\theta(T_1 + T_2 + \ldots + T_n} \;\middle|\; N = n\right] = \prod_{i=1}^n E\left[e^{\theta T_i}\right].$$

   To obtain the last equality above, we have used the fact that the T_i are mutually independent, and independent of the random variable N; hence, we can drop the conditioning on N, and replace the expectation of the product of $e^{\theta T_i}$ by the product of their expectations. Now, substituting for $M_i(\theta)$ above, we get,

$$E\left[e^{\theta T} \;\middle|\; N = n\right] = M_i(\theta)^n,$$

i.e.,

$$E\left[e^{\theta T} \;\middle|\; N\right] = M_i(\theta)^N = \begin{cases} \left(\frac{\lambda}{\lambda-\theta}\right)^N, & \text{if } \theta < \lambda, \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

Next, observe that the generating function of the random variable N is given by

$$G_N(z) := E\left[z^N\right] = \sum_{k=1}^{\infty} p(1-p)^{k-1} z^k = \begin{cases} \frac{pz}{1-(1-p)z}, & \text{if } z < 1/p, \\ +\infty, & \text{otherwise.} \end{cases}$$

Hence, on taking expectations with respect to N in (1), we obtain the moment generating function of T as

$$\begin{aligned} M_T(\theta) &:= E\left[e^{\theta T}\right] = E\left[E\left[e^{\theta T} | N\right]\right] \\ &= G_N(M_i(\theta)) = \begin{cases} \left(\frac{\lambda p}{\lambda-\theta}\right) \Big/ \left(1 - \frac{\lambda(1-p)}{\lambda-\theta}\right), & \text{if } \theta < \lambda \text{ and } \frac{\lambda}{\lambda-\theta} < \frac{1}{p}, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

Simplifying the above expression, we finally have

$$M_T(\theta) = \begin{cases} \frac{\lambda p}{\lambda p - \theta}, & \text{if } \theta < \lambda p, \\ +\infty, & \text{otherwise.} \end{cases}$$

We recognise this as the moment generating function of an Exp(λp) random variable. Hence, using the fact that there is a one-to-one correspondence between probability distributions and moment generating functions (which we shall take for granted without proof), we conclude that T is an exponential random variable with parameter λp.

5. Let $T_1, T_2, \ldots$ be the times of successive events in the Poisson process $X_t, t \geq 0$, and let $T_1^1, T_2^1, \ldots$ denote the same for the process $X_t^1, t \geq 0$. Then,

$$\begin{aligned} P(T_1^1 \geq t) &= \sum_{n=1}^{\infty} P(T_n \geq t, Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) \\ &= \sum_{n=1}^{\infty} P(Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) P(T_n \geq t | Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) \\ &= \sum_{n=1}^{\infty} P(Y_1, Y_2, \ldots, Y_{n-1} = 0, Y_n = 1) P(T_n \geq t), \end{aligned}$$

since the Y_i are independent of the Poisson process $X_t, t \geq 0$.

Now, the first i for which $Y_i = 1$ is a geometric random variable, and $T_{i+1} - T_i$ for successive i are independent Exp(λ) random variables. Hence, T_1^1 is the sum of a Geom(p) number of iid copies of an Exp(λ) random variable. Hence, by the answer to Question 4, T_1^1 is an Exp(λ_p) random variable. The same argument applies to $T_2^1 - T_1^1$, $T_3^1 - T_2^1$ and so on, which are also clearly mutually independent (due to the mutually independence of times between events in the X_t process, and the fact that the Bernoulli sequence Y_i is iid). To show that $X_t^1, t \geq 0$ is a Poisson process of rate λp, it remains only to show that $X_{t+u}^1 - X_t^1$ is independent of $X_s^1, s \leq t$. But this is obvious from the discussion above.

6. Recall that $P(t) = e^{Qt}$. Using the diagonalisation of Q, we have

$$\begin{aligned} P(t) &= e^{Qt} = I + Qt + \frac{Q^2t^2}{2!} + \frac{Q^3t^3}{3!} + \dots \\ &= I + A(Dt)A^{-1} + A\frac{(Dt)^2}{2!}A^{-1} + A\frac{(Dt)^3}{3!}A^{-1} + \dots \\ &= A\Big(I + Dt + \frac{(Dt)^2}{2!} + \frac{(Dt)^3}{3!} + \dots\Big)A^{-1} = Ae^{Dt}A^{-1}. \end{aligned}$$

The point of this is that powers of a diagonal matrix simply correspond to taking the powers of the diagonal entries, element by element. Consequently, e^{Dt} is also a diagonal matrix, with jj^{th} element equal to $e^{d_{jj}t}$.