1. The contact rates are $q_{ij} = 1$ whenever $(i, j) \in E$, and zero otherwise (for $j \neq i$). Hence, the $Q$ matrix is

$$Q = \begin{pmatrix}
     -3 & 1 & 1 & 1 \\
     1 & -2 & 1 & 0 \\
     1 & 1 & -3 & 1 \\
     1 & 0 & 1 & -2
\end{pmatrix}.$$ 

As each of the columns sum to zero, it is easy to see that $\pi = (1, 1, 1, 1)/4$ solves the global balance equations $\pi Q = 0$. Alternatively, the local balance equations are $\pi_i \cdot 1 = \pi_j \cdot 1$ whenever $(i, j) \in E$, i.e., $\pi_i = \pi_j$ if there is an edge between $i$ and $j$. Consequently, $\pi_i = \pi_j$ if there is a path between $i$ and $j$. As the graph is connected, $\pi_i = \pi_j$ for all $i$ and $j$. In other words, the uniform distribution solves the local balance equations, and hence is invariant. Moreover, $\pi$ is the unique invariant distribution as it is the only probability vector satisfying either the global or local balance equations.

Let $X(t) \in \{0, 1\}^V$ denote the vector of states of all nodes at time $t$, which evolves as a Markov process. By the results in the lecture notes, $M(t) = \pi \cdot X(t)$ is a martingale, i.e.,

$$\mathbb{E}[M(t + \tau)|X(s), s \leq t] = M(t) \text{ for all } \tau > 0 \text{ and all } t.$$ 

Note that we need to condition on the past of $X(t)$; it may not be enough just to condition on $M(s), s \leq t$.

(More precisely, the definition of a martingale has to state what we are conditioning on - and the martingale has to be measurable with respect to that - but we are going to be less formal and avoid measure-theoretic terminology. However, you do need to have the right intuition about what to condition on in order to get the martingale property.)

Using the expression for $\pi$, this says that $M(t) = \sum_{i=1}^n X_i(t)/4$ is a martingale. The random time $T$ at which consensus is reached is a stopping time, and the martingale is bounded (by 1), so we can use the optional stopping theorem to conclude that $\mathbb{E}M(T) = M(0)$. But $M(0) = 2/4$ as we start with two of the nodes in state 1, and so

$$\mathbb{E}M(T) = 1 \cdot \mathbb{P}(X(T) = 1) + 0 \cdot \mathbb{P}(X(T) = 0) = M(0) = \frac{1}{2}.$$ 

Hence, $\mathbb{P}(X(T) = 1) = 1/2$ is the probability of reaching consensus on the all-1 state.

2. (a) Suppose that at time $t$ the hub and $k-1$ leaves are in state 1, for some $k \in \{1, 2, \ldots, n-1\}$. Then, either one of the $n-k$ leaves in state 0 flips to state 1, which happens at aggregate rate $n-k$ (the total rate at which one of these $n-k$ leaves becomes active, since it then necessarily copies the hub), or the hub flips from state 1 to state 0, which
happens at rate \( (n - k)/(n - 1) \) (the probability that, when the hub becomes active, it chooses one of the leaves in state 0 to copy). Hence, in this case,

\[
\mathbb{E}[M(t + dt) - M(t)|(X(u), u \leq t)] = 1 \cdot (n - k)dt - (n - 1) \cdot \frac{n - k}{n - 1}dt = 0,
\]

eventing \( o(dt) \) terms.

The analysis is identical if the hub is in state 0, since the two states are perfectly symmetrical. Hence, we have shown that the expectation of \( M(t) \) remains constant so long as all the nodes aren’t in the same state. But once all nodes are in the same state (0 or 1), they remain in that state for ever. Hence, \( M(t) \) remains constant, deterministically, from then on; trivially, so does its expectation. Thus, we have shown that \( M(t) \) is a martingale.

(b) For the given initial state, we have \( M(0) = (n - 1) + (k - 1) = n + k - 2 \). Let \( T \) be the random time to hit one of the two absorbing states, the all-0 or all-1 states. Then \( T \) is a stopping time. Moreover, \( M(T) = (n - 1) + (n - 1) \) on the event that the all-1 state is reached at time \( T \), while \( M(T) = 0 \) on the event that the all-0 state is reached. Hence,

\[
\mathbb{E}[M(T)] = 2(n - 1) \cdot P_k(X(T) = 1) + 0 \cdot P_k(X(T) = 0).
\]

By the Optional Stopping Theorem, \( \mathbb{E}[M(T)] = \mathbb{E}[M(0)] = M(0) \), i.e.,

\[
2(n - 1) \cdot P_k(X(T) = 1) = n + k - 2,
\]

which implies that

\[
P_k(X(T) = 1) = \frac{n + k - 2}{2(n - 1)}.
\]

3. (a) Each leaf copies the hub at rate 1. The hub copies each leaf at rate \( 1/(n - 1) \), adding up to rate 1 for activity. Hence, the random walk has rate 1 of moving from a leaf to the hub, and rate \( 1/(n - 1) \) of moving from the hub to any given leaf.

(b) Let \( X_t \) denote the distance between the two random walks at time \( t \). From the answer to the last part, we see that \( X_t \) is a Markov process with transition rates \( q_{21} = 2 \), \( q_{12} = (n - 2)/(n - 1) \) and \( q_{10} = 1 + 1/(n - 1) \). All other off-diagonal transition rates are zero. Note that for \( X_t \) to go from 2 to 1 (the event whose rate is denoted \( q_{21} \), we must have the two random walks at different leaves at time \( t \), and one of them moving to the hub during the interval \((t, t + dt)\). As either of them could move at rate 1, the overall rate is 2. Likewise, \( X_t \) goes from 1 to 2 if one walk is at the hub, one at the leaf, and the one at the hub moves to another leaf; the rate for this is the rate that the walk at hub moves to any one of \( n - 2 \) leaves (but not the one occupied by the other random walk). Finally, for \( X_t \) to go from 1 to 0, either the random walk at the leaf should move to the hub, or the one at the hub should move to the specific leaf occupied by the other walk.

(c) Let \( \alpha_x \) denote the expected time for \( X_t \) to hit state 0 starting in state \( x \). We will obtain simultaneous equations for the \( \alpha_x \) and solve them. Clearly, \( \alpha_0 = 0 \). To compute \( \alpha_2 \), we note that \( X_t \) spends a random \( \text{Exp}(2) \) time, with mean \( 1/2 \), in state 2 before moving to state 1, after which it needs expected time \( \alpha_1 \) to get to state 0. Hence,

\[
\alpha_2 = \frac{1}{2} + \alpha_1. \tag{1}
\]
4. (a) Suppose $Y_t$ lies between 1 and $n - 1$. Note that each particle moves at rate 1, and moves left or right with equal probability. Hence, $Y_{t+dt} - Y_t = \pm 1$ with equal probability $1 \cdot dt$ each. Hence,

$$
E[Y_{t+dt}^2 - Y_t^2 | Y_t] = \left((Y_t + 1)^2 - Y_t^2\right) dt + \left((Y_t - 1)^2 - Y_t^2\right) dt = 2dt,
$$

and so $E[Y_{t+dt}^2 - 2(t + dt) - Y_t^2 + 2t] = 0$, which establishes that $Y_t^2 - 2t$ is a martingale.

(b) Note that $Y_t^2 - 2t$ is a bounded martingale, as it is bounded by $n^2$, and that $T$ is a stopping time. Hence, by the optional stopping theorem, $E[Y_T^2 - 2T] = Y_0^2 - 0 = |i - j|^2$. But $Y_T$ is either 0 or $n$ by definition of $T$, so we also have

$$
E[Y_T^2 - 2T] = E[Y_T^2] - 2E[Y_T] = n^2P(Y_T = n) + 0 \cdot P(Y_T = 0) - 2E[T] = n^2P(Y_T = n) - 2E[T].
$$

Equating this to $|i - j|^2$, and using the given expression, $P(Y_T = n) = |i - j|/n$, we get $E[T] = |i - j|(n - |i - j|)/2$. 

Remark. The bounding technique used in this homework problem was for illustration. It yields a very loose bound. A more careful analysis, using Chernoff bounds, yields an upper bound on the time to consensus which is logarithmic in $n$ rather than linear in $n$. 