UNIVERSITY OF BRISTOL

School of Mathematics

INTRODUCTION TO QUEUEING NETWORKS

MATH35800

(Paper code MATH–35800J)

January 2018   1 hours 30 minutes

This paper contains THREE questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. (a) (**10 marks**)
Let N be a Poisson random variable with mean λ, and let $X_1, X_2, \ldots$ be iid Bernoulli(p) random variables independent of N. Let $Y = X_1 + X_2 + \cdots + X_N$. Using generating functions, show that Y is a Poisson random variable with mean λp.

   (b) (**10 marks**)
Let X and Y be independent Poisson random variables with means λ and μ respectively, and let $Z = X + Y$. Using generating functions or otherwise, show that Z is Poisson with mean $\lambda + \mu$.

   (c) (**15 marks**)
A hospital operating theatre performs a random number of operations each day (including weekends), having a Poisson distribution with mean 3. The number of operations performed on different days are mutually independent. After the operation, a patient has to remain in the ward for between 1 and 5 days, with respective probabilities 0.1, 0.3, 0.3, 0.2, 0.1. The duration of stay is independent across patients, and independent of the arrival process. The ward has infinitely many beds. Patients whose operation is scheduled on a given day are admitted into the ward the previous night.

   Explain clearly why the number of patients in the ward at the *start* of a given day (while the patients due to be discharged that day are still in the ward) is a Poisson random variable, and compute its mean.

2. (**25 marks**)
Consider the following model for a slip road joining a motorway. Cars on the motorway flow past according to a Poisson process of rate λ, taking priority over the slip road. Cars on the slip road can enter the motorway only if the time interval between two cars on the motorway is at least one time unit. If the time interval is T units, then $\lfloor T \rfloor$ cars from the slip road can join the motorway during this interval, where $\lfloor T \rfloor$ denotes the integer part of T; for example, if the gap is 3.8 time units, then 3 cars can get through.

   Compute the maximum throughput achievable on the slip road, i.e., the maximum rate at which cars on this road can join the motorway, as a function of λ.

   Hint. Let T be the random length (duration) of a typical gap between cars on the motorway. Find the expected number of cars that can get through during this gap, and divide by the expected value of T.

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3. (a) Visitors arrive at the website of a popular newspaper according to a Poisson process of rate (intensity) 2 per second, and request an article. Requests are served in the order of arrival, and it takes the webserver an exponentially distributed time with mean 100ms to serve a requested article. Upon being served, the visitor spends a random length of time uniformly distributed between 100 and 300 seconds reading the content. After finishing the article, the visitor has probability 3/4 of leaving the system, and probability 1/4 of requesting a different article, independent of the past.

   i. (**15 marks**)
   What is the expected wating time between requesting an article and the request being served?

   ii. (**10 marks**)
   At a given point in time, what can you say about the mean and the distribution of the number of readers on the newspaper's website? Exclude those who are waiting to be served a requested article.

   (b) (**15 marks**)
   Landing slots at a certain airport are exactly 3 minutes long, i.e., the time between successive aircraft being given permission to land is 3 minutes. Aircraft scheduled for landing arrive into its airspace according to a Poisson process of rate 15 per hour, and are given permission to land strictly in the order of their arrival.

   What is the mean number of aircraft waiting for permission to land at a given time, excluding any that has been given permission to land but that may not yet have completed landing?

End of examination.