Introduction to Queuing Networks

Problem Sheet 3

$**$ **Please hand in solutions to questions 1 and 5 on this sheet.** $**$

1. Suppose people arrive at a bus stop individually (never in groups) according to a Poisson process of rate λ, and that buses arrive according to a Poisson process of rate μ. Buses are infinitely large and there is only one route, so that when a bus arrives, everyone waiting at the bus stop gets on to it.

   (a) Let X_t denote the number of people waiting at the bus stop at time t. Describe X_t as a continuous time Markov chain (CTMC), i.e., specify all the states and transition rates, either in the form of an arrow diagram or a transition rate matrix.

   (b) Show that the local balance equations have no solution. (Hence, this Markov chain is not reversible.)

   (c) Solve the global balance equations to find the invariant distribution of this Markov chain. What conditions on λ and μ do you need for there to be an invariant distribution? Explain this intuitively.

   (d) Compute the mean number of people waiting at the bus stop, and use Little's law to find out how long a typical customer has to wait for a bus. Is there a more direct way to reach the same answer?

2. Let $\{N_t, t \geq 0\}$ be a Poisson arrival process with rate λ, and let A be the number of arrival events in a random interval of length W, where W has Exponential(μ) distribution and is independent of the process N_t. For example, A could represent the number of arrivals to an $M/M/1$ queue (with arrival rate λ and service rate μ) during a random service period.

   By conditioning on the value of W, show that

   $$P(A = k) = \frac{\mu}{\lambda + \mu}\Big(\frac{\lambda}{\lambda + \mu}\Big)^k, \quad k = 0, 1, 2, \ldots,$$

   so that A has a Geometric distribution on the integers $0, 1, 2, \ldots$ with parameter $\mu/(\mu + \lambda)$.

3. **M/M/1 queue with balking** Sometimes customers arriving to a system may be discouraged from joining by the sight of a long queue and may balk, i.e., decide not to join the system. Consider an $M/M/1$ queue where arrivals form a Poisson process of rate λ, where service times are iid $Exp(\mu)$ random variables independent of the arrival process, and where there is a single server and infinite waiting room.

   For $i = 0, 1, 2, \ldots$, suppose that any job that arrives and finds i jobs already in the system ahead of it (including any being served) has probability $1/(1 + i)$ of actually joining the queue, and probability $i/(1 + i)$ of leaving the system right away.

(a) Write down the transition rates q_{ij} and the jump probabilities p_{ij} for the CTMC describing the number of jobs in the system. Use the transition rates to find the invariant distribution of this CTMC.

(b) Use Bayes' formula to compute the distribution of the number of customers already present in the system, as seen by a typical arrival who (i) decides to join the queue, and (ii) decides not to join the queue.

4. Consider the equilibrium behaviour of an $M/M/2$ queue with 2 servers, where the service times for both servers are independent Exp(μ) random variables and where there is infinite waiting room. Assume jobs arrive at rate λ when the systen is not empty and arrive at a different rate $\alpha \neq \lambda$ when the system is empty. Thus the times between the arrival of successive jobs are always independent Exponential random variables, but the parameter of the Exponential distribution is λ when the system is not empty and α when the system is empty.

   (a) Using the fact that the system is a birth and death process, write down an expression for the stationary probabilities $\{\pi_j; j \in S\}$ in terms of π_0, α, λ and μ and hence show that $\pi_0 = (2\mu - \lambda)/(2\mu + 2\alpha - \lambda)$.

   (b) Now let X_A denote the number of jobs in the system as seen by a random arrival. For $j = 0, 1, 2, \ldots$, find $P(X_A = j)$ in terms of π_0, α, λ and μ. Hence show that in equilibrium, the number of jobs in the system as seen by a random arrival does not have the same distribution as the stationary distribution for this system.

5. Consider a single server $M/M/1$ queue with Exp(μ) service distribution. Assume jobs arrive as a Poisson process with rate $\lambda < \mu$, and let $\rho = \lambda/\mu$.

   (a) From your notes, write down the stationary distribution $\boldsymbol{\pi}$ for the system.

   (b) Now assume the system is in equilibrium. Let $t > 0$ be a fixed time and let C^* denote the first job to arrive after t. Show that the probability C^* finds j jobs already in the system is given by

$$\begin{array}{ll} (1-\rho)(1+\rho) & \text{for} \quad j = 0 \\ (1-\rho)\rho^{j+1} & \text{for} \quad j = 1, 2, 3, \ldots \,. \end{array}$$

   Hence show that in equilibrium, the distribution of the number of jobs in the system as seen by C^* is not the same as the stationary distribution.