Elements of Linear Programming
Problem Sheet

1. Consider the following linear program:

$$
\begin{aligned}
&\text{Maximize } x_1 + 3x_2 + 2x_3 + x_4 \\
&\text{Subject to} \\
&\quad x_1 - 2x_2 + x_3 + x_4 \le 4, \\
&\quad -x_1 + 3x_2 - x_3 + 2x_4 \ge 5, \\
&\quad x_1 + x_2 + x_3 + x_4 = 10, \\
&\quad x_1 \ge 0,\ x_3 \le 0.
\end{aligned}
$$

   Formulate an equivalent standard equality linear program.

2. (a) Consider the following optimization problem.

$$
\begin{aligned}
&\text{Minimize} \quad \max_{1\le k\le p} \left| \sum_{j=1}^{n} c_{kj}x_j + d_k \right| \quad \text{(with respect to } x_1, \ldots, x_n \in \mathbb{R}) \\
&\text{Subject to} \\
&\quad \sum_{j=1}^{n} a_{ij}x_j \ge b_i, \quad \text{for } i = 1, \ldots, m, \\
&\quad x_j \ge 0, \quad \text{for } j = 1, \ldots, n. \qquad (1)
\end{aligned}
$$

   Formulate this problem as a linear programming problem.

   (b) Consider the following optimization problem.

$$
\begin{aligned}
&\text{Minimize} \quad \sum_{k=1}^{p} \left| \sum_{j=1}^{n} c_{kj}x_j + d_k \right| \quad \text{(with respect to } x_1, \ldots, x_n \in \mathbb{R}) \\
&\text{Subject to} \\
&\quad \sum_{j=1}^{n} a_{ij}x_j \ge b_i, \quad \text{for } i = 1, \ldots, m, \\
&\quad x_j \ge 0, \quad \text{for } j = 1, \ldots, n. \qquad (2)
\end{aligned}
$$

   Formulate this problem as a linear programming problem.

3. (a) Using the graphical method, solve the following linear program.

$$\text{Minimize } x+y$$
$$\text{Subject to } 2x+3y \geq 1, x-y \geq 0, x \geq 0, y \leq 0.$$

(b) Using the graphical method, solve the following linear program.

$$\text{Minimize } x+y$$
$$\text{Subject to } 2x+3y \geq 1, x-y \geq 0, x \geq 1, y \leq 2.$$

(c) Using the graphical method, solve the following linear program.

$$\text{Minimize } x+y$$
$$\text{Subject to } 2x+5y \geq 1, x-y \geq 0, x \leq 0.$$

(d) Using the graphical method, solve the following linear program.

$$\text{Minimize } x+y$$
$$\text{Subject to } -2x+y \geq 2, x-2y \leq -2, x \leq 0, y \geq 0.$$

(e) Using the graphical method, solve the following linear program.

$$\text{Maximize } 7x+6y$$
$$\text{Subject to } 7x+2y \geq 28, x+6y \geq 12, 14x+12y \leq 168.$$

4. Find all basic feasible solutions of the following linear program:

$$\text{Minimize } x_1+x_2$$
$$\text{Subject to}$$
$$\begin{aligned} x_1 + x_2 \quad + x_3 \qquad &= 6 \\ + x_2 \qquad\quad + x_4 \ &= 3 \end{aligned}$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Using the obtained result, find an optimal solution.

5. Consider the following linear program:

$$\text{Minimize } \sum_{j=1}^{n} c_j x_j$$
$$\text{Subject to}$$
$$\sum_{j=1}^{n} a_j x_j = b$$
$$x_1 \geq 0, \ldots, x_n \geq 0$$

Here, $a_1, c_1, \ldots, a_n, c_n \in \mathbb{R}$ and $b \in (0, \infty)$.
(a) Derive a simple test for checking the feasibility.
(b) Assuming that the optimal solution is finite, compute the optimal cost value.

6. Consider the following linear program:

$$\begin{aligned}
&\text{Minimize } \sum_{j=1}^{n} c_j x_j \\
&\text{Subject to} \\
&\quad \sum_{j=1}^{n} a_j x_j = b \\
&\quad \sum_{j=1}^{n} x_j = 1 \\
&\quad x_1 \geq 0, \ldots, x_n \geq 0
\end{aligned}$$

Assuming $a_1 < a_2 < \cdots < a_{n-1} < b < a_n$, check if the linear program is feasible. If it is, find the optimal cost value.

7. (a) Combining the duality principle and the graphical method, solve the following linear program:

$$\begin{aligned}
&\text{Minimize } x_1 - x_2 + x_3 \\
&\text{Subject to} \\
&\quad x_1 + 2x_2 \quad +x_3 = 3 \\
&\quad x_1 + x_2 \quad -x_3 = 1 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{aligned}$$

(b) Combining the duality principle and the graphical method, solve the following linear program:

$$\begin{aligned}
&\text{Maximize } x_1 + x_2 + x_3 \\
&\text{Subject to} \\
&\quad x_1 + 2x_2 \quad +x_3 = 1 \\
&\quad 2x_1 + x_2 \quad -x_3 = 4 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
\end{aligned}$$

(c) Combining the duality principle and the graphical method, solve the following linear program:

$$\begin{aligned}
&\text{Minimize } 2x_1 + 3x_2 + 3x_3 + 6x_4 + 4x_5 \\
&\text{Subject to} \\
&\quad 2x_1 + x_2 \quad -2x_3 + 3x_4 \quad -2x_5 = -1 \\
&\quad x_1 + 3x_2 \quad +x_3 + 2x_4 \quad +x_5 = 1 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.
\end{aligned}$$

(d) Combining the duality principle and the graphical method, solve the following linear program:

$$\begin{aligned}
&\text{Minimize } 2x_1 + 3x_2 + 5x_3 + 2x_4 + 3x_5 \\
&\text{Subject to} \\
&\quad x_1 + x_2 + 2x_3 + x_4 + 3x_5 \geq 4 \\
&\quad 2x_1 - 2x_2 + 3x_3 + x_4 + x_5 \geq 3 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.
\end{aligned}$$

8. (a) Consider the following linear program:

$$\begin{aligned}
&\text{Minimize } x_1 + 5x_2 + 2x_3 + 13x_4 \\
&\text{Subject to} \\
&\quad 5x_1 - 6x_2 + 4x_3 - 2x_4 = 0 \\
&\quad x_1 - x_2 + 6x_3 + 9x_4 = 16 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{aligned}$$

Using the duality principle, check if

$$x_1 = 0,\ x_2 = 2,\ x_3 = 3,\ x_4 = 0$$

is an optimal solution.

(b) Consider the following linear program:

$$\begin{aligned}
&\text{Minimize } -6x_1 - 6x_2 - 4x_3 \\
&\text{Subject to} \\
&\quad 4x_1 + 6x_2 + 2x_3 + x_4 = 24 \\
&\quad 4x_1 + 4x_2 + 3x_3 = 20 \\
&\quad 2x_1 + 3x_2 + x_3 = 8 \\
&\quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0
\end{aligned}$$

Using the duality principle, check if

$$x_1 = 2,\ x_2 = 0,\ x_3 = 4,\ x_4 = 8$$

is an optimal solution.