Introduction to Statistics Problem Sheet 1

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1. The Rayleigh distribution has a single parameter σ, called its scale parameter; it has density

$$f(x) = \frac{x}{\sigma^2} \exp\Big(-\frac{x^2}{2\sigma^2}\Big), \quad x \geq 0.$$

Given independent and identically distribution (iid) samples (observations) $x_1, x_2, \ldots, x_n$ drawn from this distribution,find a method of moments estimator and a maximum likelihood estimator for the scale parameter σ.

Hint. In computing the mean of the Rayleigh distribution, you might find it helpful to use the known fact that the Gaussian distribution $N(0, \sigma^2)$ with mean zero and variance σ^2 has density

$$\phi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{x^2}{2\sigma^2}\Big), \quad x \in \mathbb{R}.$$

Warning! Note that the supports of the two distributions are different. Take care of this when using the stated fact.

2. You are given a possibly biased coin and wish to determine the probability that it comes up heads. You toss it repeatedly until you see k heads. Let N be the random number of tosses required. We shall consider three different estimators of the probability p of obtaining heads. The first estimator is $\tilde{p} = k/N$.

To obtain the other two estimators, recall that the number of coin tosses X_1 required until seeing the first head has a Geometric(p) distribution, i.e.,

$$\mathbb{P}(X_1 = j) = (1-p)^{j-1}p, \; j = 1, 2, 3, \ldots.$$

Similarly, the number of coin tosses X_2 after the first head until seeing the second also has a Geometric(p) distribution, and likewise for $X_3, X_4, \ldots X_k$, which denote the number of coin tosses required for each subsequent head. Moreover, these random variables are mutually independent, and $X_1 + X_2 + \ldots + X_k = N$.

(a) Suppose we are given $X_1, X_2, \ldots, X_k$, i.e., we are told the exact sequence observed and not just the number of tosses required until the k^{th} head. Use this information to develop a Method of Moments estimator $\hat{p}_{MoME}$ and a Maximum Likelihood estimator $\hat{p}_{MLE}$ for the unknown parameter. You will need to calculate or look up the mean of a geometric distribution in order to answer this.

(b) Notice that these estimators depend only on N and not on the values of the individual X_j's. Why do you think this is? Can you justify your answer?

(c) Compute the bias and mean square error of each of the three estimators above. This will require you to compute or look up the second moment of a geometric distribution.

3. Let U be a uniform distribution on $[0, a]$, where a is unknown. You are given iid observations $x_1, x_2, \ldots, x_n$ drawn from this distribution. Obtain a method of moments estimator $\hat{a}_{MoME}$ and a maximum likelihood estimator $\hat{a}_{MLE}$ for the unknown parameter, in terms of the data.

   Compute the bias and the MSE of each of these two estimators.

4. This problem is motivated by the Flajolet-Martin algorithm (P. Flajolet and N. Martin, "Probabilistic counting algorithms for database applications", *J. Comp. and Sys. Science*, 1985).

   Suppose X has a shifted exponential distribution, with density

$$f_X(x) = \begin{cases} 0, & x < \alpha, \\ \lambda e^{-\lambda(x-\alpha)}, & x \geq \alpha, \end{cases}$$

   where the shift parameter α and scale parameter λ are both unknown. We are given iid observations $x_1, x_2, \ldots, x_n$ from this distribution.

   Compute a method of moments estimator and a maximum likelihood estimator for the unknown parameters.

   Hint. Notice that the density of X is the density of the random variable $Y + \alpha$, where Y is an Exp(λ) random variable. You may use the fact that an Exp(λ) distribution has mean $1/\lambda$ and variance $1/\lambda^2$.

5. A certain weak light source emits a random number of photons with a Poisson(λ) distribution over an observation period. In other words, if X denotes a random variable with this distribution, then

$$\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots$$

   We take independent observations over n observation periods using one of two devices. A *counter* counts the number of photons emitted, while a *detector* only detects if at least one photon was emitted over the observation window. In other words, the detector output is described by the random variable Y, which takes the value 0 if $X = 0$, and 1 if $X \geq 1$.

   (a) Given n iid observations from each of these devices, derive maximum likelihood estimators of the unknown parameter λ. In each case, work out the MSE of your estimator.

   (b) In each case, how many observations do you need in order to ensure that the MSE is smaller than $0.01\lambda^2$, i.e., the relative error of the estimate is smaller than 10%? If it costs \$1 per observation with the detector and \$5 per observation with the counter, which would you rather use?