Environmental and Industrial Flows **Dr A.J. Hogg**
Example Sheet 1 **October 2006**

Dimensional analysis and scaling

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1. (a) A particle of mass m is dropped from rest and accelerates with gravitational acceleration g. After it has fallen a distance h it has attained speed u. Use dimensional analysis to express u in terms of m, g and h.

   (b) Suppose instead that the particle is released with an initial speed v and that its motion is resisted by a drag force of magnitude k multiplied by the square of the velocity of the particle, where k is a constant. What are the dimensions of k? Use dimensional analysis to propose an expression for u in terms of k, m, h, v and g.

   (c) Now solve the full problem by integrating the equations of motion and verify that the predicted expression for u is consistent with the exact solution.

2. Consider a boat propelled by N rowers (where $N = 1, 2, 4$ or 8). Assume that each rower is of the same weight and that each rows with the same power. Boats of different sizes are taken to be geometrically similar and drag arises from the friction of the water. From these assumptions it can be shown that the speed of the boat V depends on the number of rowers N, the power delivered per rower P, the volume of water displaced by each rower G and the density of water ρ. Using dimensional analysis show that V must be proportional to $N^{1/9}$.

3. A spherical rain drop of mass $m(t)$ falls at velocity $v(t)$ under gravity with negligible air resistance through a cloud of water vapour. The rate of accretion is given by

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho_v \pi a^2 v,$$

   where ρ_v is the density of water vapour and a is the radius of the drop, such that the mass $m = \frac{4}{3}\rho_l \pi a^3$, where ρ_l is the density of water in the droplet. Taking $a(0) = v(0) = 0$ find how the velocity of the drop and how it grows in time.

   [Hint: Use dimensional analysis to formulate expressions, which should then be substituted into the equations of motion to find the unknown coefficients.]

Plume theory

4. Consider the motion of an axisymmetric turbulent jet generated by the injection of fluid through a uniform ambient, composed of identical fluid. The jet has initial volume flux πQ_0 and initial momentum flux πM_0. Develop a model of the motion using equations which model conservation of mass and momentum and find solutions for the width and velocity of the jet. Show that when $z \gg Q_0/M_0^{1/2}$, the jet behaves as if it were solely generated by the momentum flux M_0. Give a physical interpretation of this result.

5. Consider the motion of a line plume which develops from a line source of buoyancy, f_0 in a linearly stratified environment.

   (a) On the assumption that the stratification is negligible, use dimensional analysis to establish expressions for how the half-width of the plume, b, the vertical velocity, w and the reduced gravity, g' depend on the buoyancy flux per unit width f_0 and height z.

   (b) Formulate conservation equations for the motion of the plume. (Use 'top' hat profiles and include ambient stratification.)

   (c) Find solutions for the motion in the limit of vanishing stratification for a 'pure' line plume.

   (d) When the stratification is non-vanishing, use dimensional analysis to identify the height to which the line plume rises. *Construct numerical solutions to the full equations.

6. (a) Write down the equations of motion for an axisymmetric, turbulent buoyant plume rising through a uniform environment. The initial buoyancy, momentum and volume fluxes are denoted, πB_0, M_0 and Q_0, respectively. Establish the following relationship between the volume and momentum flux

$$\frac{8\epsilon M(z)^{5/2}}{5\sqrt{\pi}B_0 Q_0^2} + c = \frac{Q(z)^2}{Q_0^2},$$

   where ϵ is the entrainment coefficient and c is a dimensionless constant which is to be determined in terms of M_0, Q_0 & B_0. Evaluate c in the regimes: (i) $M_0^{5/2} \gg B_0 Q_0^2$ and (ii) $M_0^{5/2} \ll B_0 Q_0^2$.
   Interpret the physical significance of the cases in which $c = 0$, $c > 0$ and $c < 0$.

   (b) When $c = 0$, show that

$$\frac{Q(z)}{Q_0} = \left[1 + \frac{6}{5}\left(\frac{5\epsilon^4\pi^3 B_0}{8Q_0^3}\right)z\right]^{5/3},$$

   and establish that $Q(z)$ becomes independent of Q_0 whne $z \gg z_s$, where z_s is to be identified.

   (c) Show that the radius of the plume, $b(z)$, is given by

$$\frac{\mathrm{d}b}{\mathrm{d}z} = 2\epsilon - \frac{4\epsilon}{5}\frac{Q^2}{Q^2 - cQ_0^2}.$$

   Thus deduce how that radius depends upon z, close to the source when $|c| \ll 1$.
   Show also that $\mathrm{d}b/\mathrm{d}z \to 6\epsilon/5$ in the far-field.

   (d) Now consider $\mathrm{d}b/\mathrm{d}z$ in the regimes (i) $M_0^{5/2} \gg B_0 Q_0^2$ and (ii) $M_0^{5/2} \ll B_0 Q_0^2$, deducing the behaviour close to the source and in the far-field.