Unlikely Connections

Random Matrices and the Riemann Zeta Function

Amy Mason University of Bristol

University of BRISTOL

What's the Riemann Zeta Function?

The Riemann zeta function is defined for $s \in C$ as the analytic continuation of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

This has an infinite number of zeros with $0 \leqslant \Re(s) \leqslant 1$, which we call the critical strip.
The Riemann Hypothesis says all zeros other than the trivial zeros have $\Re(s) = \frac{1}{2}$, falling on the critical line in the center of the critical strip.[1]
The Riemann Hypothesis is very hard to prove - over 100 years have passed since Riemann made his hypothesis.
We know that that infinitely many of the zeros, including the first 10^{13} zeros, are on $\Re(s) = \frac{1}{2}$ line - but this still leaves an infinite number of zeros to check. [2][3]

Pair Correlation

What if we stop looking at where the zeros are within the critical strip and instead look at how they are distributed up the critical line?
We know they get denser higher up the critical, but it's more interesting to look at the fluctuations about the mean density. So we define the unfolded zeroes - if t_n is the height of the n^{th} Riemann zero above the y-axis, then $\gamma_n = \frac{t_n}{2\pi} \log\left(\frac{t_n}{2\pi}\right)$ is the height of the n^{th} unfolded Riemann zero - so that the unfolded zeros have density of 1.
A good statistic to look at is the Pair Correlation function at height T. We use a well behaved test function f and define

$$F_\zeta(T) = \frac{1}{T} \sum_{\substack{i \neq j \\ \gamma_i, \gamma_j \leqslant T}} f(\gamma_i - \gamma_j).$$

We can think of this as sampling the test function at points corresponding to the distances between the Riemann zeros.

The Zeros

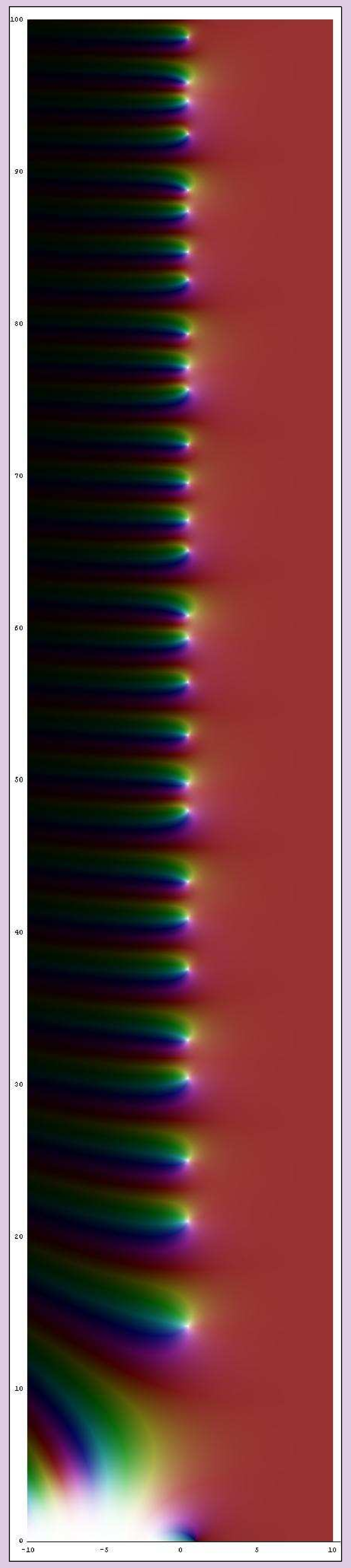

Figure 1: The Riemann zeta function about the critical line. Colour indicates argument and brightness indicates the absolute value of the function. The white spots are the zeros of the Riemann zeta function. [4]

Montgomery's Conjecture

In 1973, Montgomery conjectured [5]

$$\lim_{T \to \infty} F_\zeta(T) = \int_{\infty}^{\infty} f(x) \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right) dx$$

and proved that the conjecture held for test functions with limited support.
Dyson pointed out to him that this was exactly the density of the pair correlation of eigenvalues of random unitary matrices.

Wait, what are random matrices?

A Unitary matrix is a n by n complex valued matrix, A, such that $A^*A = I_n$ where A^* is the conjugate transpose of A. We take $U(N)$, the group of unitary matrices and equip it with a measure, called the Haar measure. We can think of the random matrices as a matrix-valued random variable, with the probability function determined by the Haar measure.
We can find the eigenvalues $e^{i\theta_1}, \cdots, e^{i\theta_n}$ of these matrices. These are n points on the unit circle.

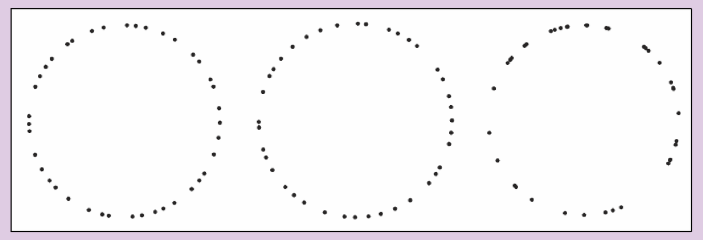

Figure 2: Eigenvalues of a random matrix size 40; 40 consecutive zeros of $\zeta(s)$ scaled to wrap once around the circle; 40 randomly chosen points on the unit circle.

We imagine unwrapping the circle into a straight line, then stretching the line to get unit density, and look at their pair correlation function

$$F_{U(N)}(N) = \frac{1}{N} \int_{U(N)} \sum_{\substack{i \neq j \\ \theta_i, \theta_j = 1}}^{N} f\left(\frac{N\theta_i}{2\pi} - \frac{N\theta_j}{2\pi}\right) dHaar.$$

It turns out that

$$\lim_{N \to \infty} F_{U(N)}(N) = \int_{-\infty}^{\infty} f(x) \left(1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2\right) dx$$
$$= \int_{-\infty}^{\infty} f(x) (R_2(x)) \, dx.$$

Are they a match?

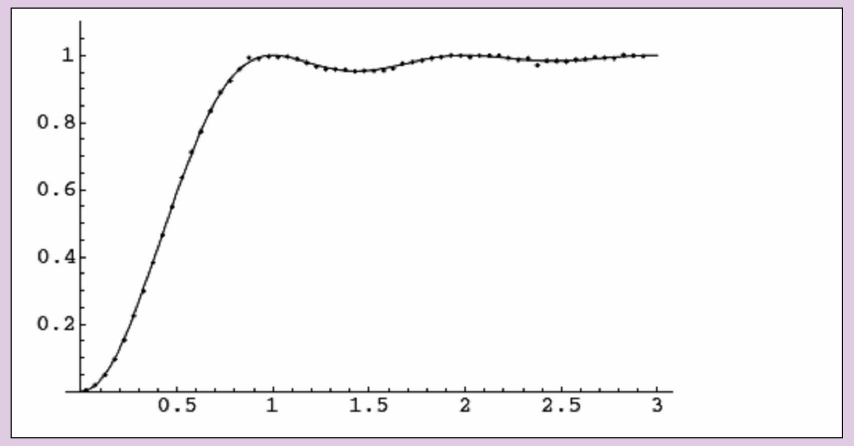

Figure 3: The pair correlation function of 10^6 zeros of the Riemann zeta function at height of the 10^{20th} zero (dots) and the pair correlation function $R_2(x)$ (smooth curve) of the eigenvalues of matrices in $U(N)$ as $N \to \infty$. The x-axis is re-scaled frequency and the y-axis is the difference between zeros or eigenvalues. [6]

We can see that infinitely high up the critical line the unfolded Riemann zeros behave like the eigenvalues of infinitely large random matrices from $U(N)$. A similar pattern is found for n-correlation. Even the moments of the Riemann zeta function can be modeled with random matrices. [7] [8]
It is clear that random matrices are a good place to start in trying to understand the Riemann zeta function.

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