SIMPLE ZEROS OF DEGREE 2 L-FUNCTIONS

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ABSTRACT. We prove that the complete L-functions of classical holomorphic newforms have infinitely many simple zeros.

1. INTRODUCTION

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$ with corresponding L-function $\Lambda(s,\pi)$. The Grand Riemann Hypothesis (GRH) and Grand Simplicity Hypothesis (GSH) predict that the zeros of $\Lambda(s,\pi)$ lie on the line $\Re(s)=\frac{1}{2}$ and are simple, apart from at most one multiple zero if π is associated to a geometric motive (cf. the BSD conjecture). These conjectures have not yet been shown to hold for a single example, and most partial evidence in their favor has been for $n=1$, i.e. the Dirichlet L-functions. In particular, until recently, the only cuspidal representation for $n>1$ for which $\Lambda(s,\pi)$ was known to have infinitely many simple zeros was the one associated to the Ramanujan Δ modular form, which is a theorem of Conrey and Ghosh [4] from 1988.

As Conrey and Ghosh remark in their paper, most of their arguments would apply to any degree 2 L-function, but they were unable to conclude the proof without assuming *a priori* the existence of at least one simple zero (which they verified directly for the L-function associated to Δ). In this paper, we analyze their method from a structural point of view, along the lines of [2] and [7], to prove the following:

Theorem 1. *Let $f\in S_k(\Gamma_1(N))^{\mathrm{new}}$ be a normalized Hecke eigenform of arbitrary weight and level. Then the complete L-function $\Lambda_f(s)=\int_0^\infty f(iy)y^{s-1}\,dy$ has infinitely many simple zeros.*

As our proof will show, a lack of simple zeros leads to inconsistencies unless the local L-factor of $\Lambda_f(s)$ is a square at every unramified prime (which cannot happen for holomorphic modular forms). In effect, we establish a connection (albeit a very loose one) between the zeros of the global L-function and those of its local factor polynomials.

Recently, Cho [3] has generalized [4] to prove that the L-functions of the first few Maass cusp forms of level 1 have infinitely many simple zeros. Our proof could be modified in an analogous fashion to extend Theorem 1 to all cuspidal Maass newforms. Moreover, the assumption that f is a cusp form is also unnecessary, so in fact the method could be generalized to show that if χ_1 and χ_2 are primitive Dirichlet characters and $t\in\mathbb{R}$ then $\Lambda(s,\chi_1)\Lambda(s+it,\chi_2)$ has infinitely many simple zeros unless $\chi_1=\chi_2$ and $t=0$. However, stronger results of this type may be obtained by other methods, e.g. [5].

Note that Conrey and Ghosh's result for $f=\Delta$ is a bit stronger than the conclusion of Theorem 1 for that case. Precisely, if $N_f^s(T)$ denotes the number of simple zeros of $\Lambda_f(s)$ with imaginary part in $[0,T]$, they showed that for every $\varepsilon>0$, the inequality $N_\Delta^s(T)\geq T^{\frac{1}{6}-\varepsilon}$ holds for some arbitrarily large values of T. With Theorem 1 in hand, it seems likely that

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their proof of this estimate would generalize at least to all eigenforms of level 1. However, in this paper we content ourselves with the qualitative statement of Theorem 1.

Finally, we remark that concurrent work of Milinovich and Ng [8] also establishes Theorem 1, assuming GRH. Although their proof is conditional, it yields the much better quantitative estimate $N_f^s(T) \geq T(\log T)^{-\varepsilon}$ for any fixed $\varepsilon > 0$ and all sufficiently large T.

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Notation. Let f be as in the statement of Theorem 1, and let ξ denote its nebentypus character. Let

$$L_f(s) = \sum_{n=1}^{\infty} a_f(n) n^{-s} = \prod_p \frac{1}{1 - a_f(p)p^{-s} + \xi(p)p^{k-1-2s}}$$

be the finite L-function of f, and $\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s)$ the completed version. Then we have the functional equation

$$\Lambda_f(s) = \epsilon N^{\frac{k}{2}-s} \Lambda_{\bar{f}}(k-s), \tag{1}$$

where $\bar{f} \in S_k(\Gamma_0(N), \overline{\xi})$ is the dual of f, and $\epsilon \in \mathbb{C}$ is the root number. We define

$$D_f(s) = L_f(s) \frac{d^2}{ds^2} \log L_f(s) = \sum_{n=1}^{\infty} c_f(n) n^{-s}.$$

Note that $D_f(s)$ continues meromorphically to $\mathbb{C}$, with poles precisely at the simple zeros of $L_f(s)$ (including the trivial zeros $s = 0, -1, -2, \ldots$).

Next, for any $\alpha \in \mathbb{Q}^\times$, we define the additive twists

$$L_f(s, \alpha) = \sum_{n=1}^{\infty} a_f(n) e(\alpha n) n^{-s} \quad \text{and} \quad D_f(s, \alpha) = \sum_{n=1}^{\infty} c_f(n) e(\alpha n) n^{-s}.$$

By Deligne's bound $|a_f(p)| \leq 2p^{\frac{k-1}{2}}$, we see that each of these is holomorphic for $\Re(s) > \frac{k+1}{2}$. Moreover, it follows from [1, Prop. 3.1] that $L_f(s, \alpha)$ continues to an entire function. One could similarly prove that $D_f(s, \alpha)$ has meromorphic continuation to $\mathbb{C}$ for every α, but it turns out to be enough for our purposes to consider $\alpha = 1/q$, where q is a prime number not dividing N. In this case, we have the following expansion of the exponential function in terms of Dirichlet characters:

$$e\left(\frac{n}{q}\right) = 1 - \frac{q}{q-1}\chi_0(n) + \frac{1}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \tau(\overline{\chi})\chi(n),$$

where $\chi_0 \pmod{q}$ is the trivial character, the sum ranges over all non-trivial $\chi \pmod{q}$, and $\tau(\overline{\chi})$ denotes the Gauss sum of $\overline{\chi}$. Multiplying both sides by $c_f(n)n^{-s}$ and summing over n, we thus see that

$$D_f\left(s, \frac{1}{q}\right) = D_f(s) - \frac{q}{q-1} D_f(s, \chi_0) + \frac{1}{q-1} \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} \tau(\overline{\chi}) D_f(s, \chi),$$

where, for each χ, $D_f(s,\chi)$ denotes the multiplicative twist

$$D_f(s,\chi)=\sum_{n=1}^{\infty} c_f(n)\chi(n)n^{-s}.$$

By the known non-vanishing results for automorphic L-functions [6], all poles of $D_f(s)/\Gamma(s)$ and $D_f(s,\chi)/\Gamma(s)$ for $\chi\neq\chi_0$ are confined to the critical strip $\left\{s\in\mathbb{C}:\Re(s)\in\left(\frac{k-1}{2},\frac{k+1}{2}\right)\right\}$. On the other hand, from the formula

$$\sum_{n=1}^{\infty} a_f(n)\chi_0(n)n^{-s}=(1-a_f(q)q^{-s}+\xi(q)q^{k-1-2s})L_f(s),$$

it follows that $D_f(s,\chi_0)$ has a pole at every simple zero of the local Euler factor polynomial $1-a_f(q)q^{-s}+\xi(q)q^{k-1-2s}$, except possibly at $s=0$ when $k=1$. By Deligne, the zeros of this polynomial occur on the line $\Re(s)=\frac{k-1}{2}$, and they are simple if and only if the polynomial is not a square. By the above, we see that $D_f(s,1/q)$ inherits these poles when they occur.

2. Proof of Theorem 1

The main tool used in the proof is the following proposition, whose proof we defer until the final section.

Proposition 2. *Suppose that $\Lambda_f(s)$ has at most finitely many simple zeros. Then, for any $\alpha\in\mathbb{Q}^\times$ and $M\in\mathbb{Z}_{\geq 0}$,*

$$D_f(s,\alpha)-\epsilon(i\,\mathrm{sgn}(\alpha))^k(N\alpha^2)^{s-\frac{k}{2}}\sum_{m=0}^{M-1}m!\left(\frac{iN\alpha}{2\pi}\right)^m\binom{s+m-1}{m}\binom{s+m-k}{m}\cdot D_{\bar f}\left(s+m,-\frac{1}{N\alpha}\right)\tag{2}$$

continues to a holomorphic function for $\Re(s)>\frac{k+1}{2}-M$.

From now on we will assume that $\Lambda_f(s)$ has at most finitely many simple zeros and attempt to reach a contradiction. To that end, let M be a positive integer and q a prime not dividing N. By Dirichlet's theorem, there are distinct primes $q_1,\ldots,q_M\in q+N\mathbb{Z}$, and it follows that $D_{\bar f}(s,-q_j/N)=D_{\bar f}(s,-q/N)$ for every j. Thus, applying Prop. 2 with $\alpha=1/q_j$, we obtain that

$$\left(\frac{N}{q_j^2}\right)^{\frac{k}{2}-s}D_f\left(s,\frac{1}{q_j}\right)-\epsilon i^k\sum_{m=0}^{M-1}m!\left(\frac{iN}{2\pi q_j}\right)^m\binom{s+m-1}{m}\binom{s+m-k}{m}D_{\bar f}\left(s+m,-\frac{q}{N}\right)\tag{3}$$

is holomorphic for $\Re(s)>\frac{k+1}{2}-M$.

Next let $m_0\in\mathbb{Z}$ with $0\leq m_0<M$. By the Vandermonde determinant, there are numbers $c_1,\ldots,c_M\in\mathbb{Q}$ such that

$$\sum_{j=1}^{M}c_jq_j^{-m}=\begin{cases}1 & \text{if } m=m_0,\\ 0 & \text{if } m\neq m_0\end{cases}\quad\text{for every } m\in\mathbb{Z}\cap[0,M).$$

Multiplying (3) by $-c_j$, summing over j and replacing s by $s-m_0$, we find that

$$\epsilon i^k m_0!\left(\frac{iN}{2\pi}\right)^{m_0}\binom{s-1}{m_0}\binom{s-k}{m_0}D_{\bar{f}}\Big(s,-\frac{q}{N}\Big)-\sum_{j=1}^{M}c_j\left(\frac{N}{q_j^2}\right)^{\frac{k}{2}+m_0-s}D_f\left(s-m_0,\frac{1}{q_j}\right)$$

is holomorphic for $\Re(s)>m_0+\frac{k+1}{2}-M$. This establishes the meromorphic continuation of $D_{\bar{f}}(s,-q/N)$ to that region. Moreover, since $D_f(s,1/q_j)$ is holomorphic on $\{s\in\mathbb{C}:\Re(s)<\frac{k-1}{2}\}\setminus\mathbb{Z}$ for each j, we see that $D_{\bar{f}}(s,-q/N)$ is holomorphic on $\{s\in\mathbb{C}:\Re(s)\in(m_0+\frac{k+1}{2}-M,m_0+\frac{k-1}{2})\}\setminus\mathbb{Z}$. Thus, choosing $m_0=2$ and M arbitrarily large, we find that $D_{\bar{f}}(s,-q/N)$ has meromorphic continuation to $\mathbb{C}$, with poles possible only at integer points.

Hence, applying Prop. 2 again with $\alpha=1/q$ and $M=2$, we learn that $D_f(s,1/q)$ can only have poles at integer points. However, we have already seen that $D_f(s,1/q)$ has a pole at every simple zero (except possibly $s=0$) of the local Euler factor polynomial $1-a_f(q)q^{-s}+\xi(q)q^{k-1-2s}$. This polynomial, in turn, has infinitely many simple zeros along the line $\Re(s)=\frac{k-1}{2}$ if and only if $|a_f(q)|<2q^{\frac{k-1}{2}}$. By the Rankin–Selberg method, the average value of $|a_f(q)|^2/q^{k-1}$ is 1, so such primes q exist in abundance. This concludes the proof of Theorem 1.

3. Proof of Proposition 2

Let $\Delta_f(s)=(2\pi)^{-s}\Gamma(s)D_f(s)$. Taking the logarithm of (1) and differentiating twice, we find

$$\psi'(s)+\frac{d^2}{ds^2}\log L_f(s)=\psi'(k-s)+\frac{d^2}{ds^2}\log L_{\bar{f}}(k-s),$$

where $\psi(s)=\frac{\Gamma'}{\Gamma}(s)$ is the digamma function. Thus, it follows that

$$\Delta_f(s)+\Lambda_f(s)(\psi'(s)-\psi'(k-s))=\epsilon N^{\frac{k}{2}-s}\Delta_{\bar{f}}(k-s). \tag{4}$$

Next, since $\Lambda_f(s)$ has at most finitely many simple zeros, there is a rectangle $\mathcal{C}$ contained within the critical strip $\{s\in\mathbb{C}:\Re(s)\in(\frac{k-1}{2},\frac{k+1}{2})\}$ which encloses all simple zeros. For $z\in\mathbb{H}=\{z\in\mathbb{C}:\Im(z)>0\}$, we define

$$F(z)=\sum_{n=1}^{\infty}c_f(n)e(nz),\quad \overline{F}(z)=\sum_{n=1}^{\infty}c_{\bar{f}}(n)e(nz),$$

$$A(z)=\frac{1}{2\pi i}\int_{\Re(s)=k-\frac{1}{2}}\big(\psi'(s)+\psi'(s+1-k)\big)\Lambda_f(s)(-iz)^{-s}\,ds,$$

and

$$B(z)=\frac{1}{2\pi i}\int_{\mathcal{C}}\Delta_f(s)(-iz)^{-s}\,ds+\frac{1}{2\pi i}\int_{\Re(s)=k-\frac{1}{2}}\frac{\pi^2}{\sin^2(\pi s)}\Lambda_f(s)(-iz)^{-s}\,ds.$$

Here $\mathcal{C}$ is given counter-clockwise orientation, and $(-iz)^{-s}$ is defined as $e^{-s\log(-iz)}$ using the principal branch of the logarithm.

These functions are related as follows:

Lemma 3. *We have*

$$F(z)+A(z)=\epsilon(-i\sqrt{N}z)^{-k}\overline{F}\left(-\frac{1}{Nz}\right)+B(z) \tag{5}$$

for all $z\in\mathbb{H}$.

Proof. By Mellin inversion, we have

$$F(z) = \frac{1}{2\pi i} \int_{\Re(s)=\frac{k}{2}+1} \Delta_f(s)(-iz)^{-s}\, ds$$

and

$$\epsilon(-i\sqrt{N}z)^{-k}\overline{F}\left(-\frac{1}{Nz}\right) = \frac{\epsilon N^{k/2}}{2\pi i} \int_{\Re(s)=\frac{k}{2}+1} \Delta_{\bar{f}}(s)(-iNz)^{s-k}\, ds.$$

Since $\Lambda_f(s)$ has at most finitely many simple zeros, there is a $\delta > 0$ such that $\Delta_{\bar{f}}(s)$ is holomorphic for $\Re(s) > \frac{k+1}{2} - \delta$. Moreover, it follows from the Phragmén–Lindelöf convexity principle that for any fixed z, $\Delta_{\bar{f}}(s)(-iNz)^{s-k}$ decays rapidly as $|\Im(s)| \to \infty$ in any fixed vertical strip. Hence, we may shift the contour of the last line to $\Re(s) = \frac{k+1-\delta}{2}$ and apply (4) to obtain

(6)

$$\begin{aligned}\frac{\epsilon N^{k/2}}{2\pi i} \int_{\Re(s)=\frac{k+1-\delta}{2}} \Delta_{\bar{f}}(s)(-iNz)^{s-k}\, ds &= \frac{\epsilon N^{k/2}}{2\pi i} \int_{\Re(s)=\frac{k-1+\delta}{2}} \Delta_{\bar{f}}(k-s)(-iNz)^{-s}\, ds \\ = \frac{1}{2\pi i} \int_{\Re(s)=\frac{k-1+\delta}{2}} \Delta_f(s)(-iz)^{-s}\, ds &+ \frac{1}{2\pi i} \int_{\Re(s)=\frac{k-1+\delta}{2}} \Lambda_f(s)[\psi'(s) - \psi'(k-s)](-iz)^{-s}\, ds.\end{aligned}$$

Note that

$$\frac{1}{2\pi i} \int_{\Re(s)=\frac{k}{2}+1} \Delta_f(s)(-iz)^{-s}\, ds - \frac{1}{2\pi i} \int_{\Re(s)=\frac{k-1+\delta}{2}} \Delta_f(s)(-iz)^{-s}\, ds = \frac{1}{2\pi i} \int_{\mathcal{C}} \Delta_f(s)(-iz)^{-s}\, ds,$$

which is the first term of $B(z)$. Next, since $\psi'(s) - \psi'(k-s)$ is holomorphic for $\Re(s) \in (0, k)$, we may shift the contour of the last integral in (6) to $\Re(s) = k - \frac{1}{2}$. Using the reflection formula $\psi'(1-s) + \psi'(s) = \pi^2/\sin^2(\pi s)$, we have

$$\psi'(s) - \psi'(k-s) = \psi'(s) + \psi'(s+1-k) - \frac{\pi^2}{\sin^2(\pi s)}.$$

This yields $A(z)$ and the remaining term of $B(z)$. $\square$

Now, the main idea of the proof of Prop. 2 is to compute $(2\pi)^s/\Gamma(s)$ times the Mellin transform of both sides of (5) along the line $\Re(z) = \alpha \in \mathbb{Q}^\times$. For $F(z)$, we have

$$\begin{aligned}\frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty F(\alpha+iy)y^s \frac{dy}{y} &= \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty \sum_{n=1}^\infty c_f(n)e(\alpha n)e^{-2\pi n y} y^s \frac{dy}{y} \\ &= \sum_{n=1}^\infty c_f(n)e(\alpha n)n^{-s} = D_f(s, \alpha).\end{aligned} \tag{7}$$

Lemma 4. *For any $\alpha \in \mathbb{Q}^\times$,*

$$\frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty A(\alpha+iy)y^s \frac{dy}{y}$$

continues to an entire function of s.

Proof. Set $\Phi(s)=\psi'(s)+\psi'(s+1-k)$. From the identity $\psi'(s)=\int_1^\infty \frac{\log x}{x-1}x^{-s}\,dx$, we get the integral representation $\Phi(s)=\int_1^\infty \phi(x)x^{-s}\,dx$ for $\Re(s)>k-1$, where $\phi(x)=\frac{(x^{k-1}+1)\log x}{x-1}$. Hence

$$\begin{aligned}\Phi(s)\Gamma(s)&=\int_1^\infty \phi(x)\int_0^\infty e^{-y}(y/x)^s\frac{dy}{y}\,dx=\int_1^\infty \phi(x)\int_0^\infty e^{-xy}y^s\frac{dy}{y}\,dx\\&=\int_0^\infty\int_1^\infty \phi(x)e^{-xy}\,dx\,y^s\frac{dy}{y}.\end{aligned}$$

Therefore, by Mellin inversion,

$$A(z)=\frac{1}{2\pi i}\int_{\Re(s)=k+1}\Phi(s)\Gamma(s)\sum_{n=1}^\infty a_f(n)(-2\pi inz)^{-s}\,ds=\sum_{n=1}^\infty a_f(n)\int_1^\infty \phi(x)e(nxz)\,dx.$$

Specializing to $z=\alpha+iy$, we get

$$A(\alpha+iy)=\sum_{n=1}^\infty a_f(n)\int_1^\infty \phi(x)e(\alpha nx)e^{-2\pi nxy}\,dx,$$

so that

$$\begin{aligned}\int_0^\infty A(\alpha+iy)y^s\frac{dy}{y}&=\sum_{n=1}^\infty a_f(n)\int_1^\infty \phi(x)e(\alpha nx)\int_0^\infty e^{-2\pi nxy}y^s\frac{dy}{y}\,dx\\&=\sum_{n=1}^\infty a_f(n)(2\pi n)^{-s}\Gamma(s)\int_1^\infty \phi(x)e(\alpha nx)x^{-s}\,dx.\end{aligned}$$

For $j=0,1,2,\ldots$, define functions $\phi_j=\phi_j(x,s)$ recursively by

$$\phi_0=\phi,\quad \phi_{j+1}=x\frac{\partial\phi_j}{\partial x}-(s+j)\phi_j.$$

Then, by integration by parts,

$$\int_1^\infty \phi_j(x,s)e(\alpha nx)x^{-s-j}\,dx=-\frac{e(\alpha n)\phi_j(1,s)}{2\pi i\alpha n}-\frac{1}{2\pi i\alpha n}\int_1^\infty \phi_{j+1}(x,s)e(\alpha nx)x^{-s-j-1}\,dx.$$

Applying this iteratively m times, we find

$$\begin{aligned}\int_1^\infty \phi(x)e(\alpha nx)x^{-s}\,dx&=e(\alpha n)\sum_{j=0}^{m-1}\frac{\phi_j(1,s)}{(-2\pi i\alpha n)^{j+1}}\\&\quad+(-2\pi i\alpha n)^{-m}\int_1^\infty \phi_m(x,s)e(\alpha nx)x^{-s-m}\,dx.\end{aligned}$$

Substituting this back into the above, we have

$$\begin{aligned}\frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty A(\alpha+iy)y^s\frac{dy}{y}&=\sum_{j=0}^{m-1}\frac{\phi_j(1,s)}{(-2\pi i\alpha)^{j+1}}L_f(s+j+1,\alpha)\\&\quad+(-2\pi i\alpha)^{-m}\sum_{n=1}^\infty\frac{a_f(n)}{n^{s+m}}\int_1^\infty \phi_m(x,s)e(\alpha nx)x^{-s-m}\,dx.\end{aligned}$$

Each of the terms in the sum over j continues to an entire function of s. On the other hand, it is straightforward to prove that $\phi_m(x,s) \ll_m (1+|s|)^m x^{k-1}$. Thus, the final sum over n is holomorphic for $\Re(s) > k-m$. Taking m arbitrarily large establishes the lemma. $\square$

Lemma 5. *Let $\alpha \in \mathbb{Q}^\times$ and $z = \alpha + iy$ for some $y \in \left(0, \frac{|\alpha|}{4}\right]$. Then*

$$\begin{aligned}
&\epsilon(-i\sqrt{N}z)^{-k}\overline{F}\left(-\frac{1}{Nz}\right)\\
&= O_{\alpha,M}(y^{M-\lfloor\frac{k+3}{2}\rfloor}) + \epsilon N^{-\frac{k}{2}} \sum_{m=0}^{M-1} \frac{(-i\alpha)^{-m-k}}{2\pi i} \int_{\Re(s)=\frac{k}{2}+1} \binom{s+m-k}{m} \left(\frac{N\alpha^2}{2\pi}\right)^{s+m}\\
&\qquad \cdot \Gamma(s+m) D_{\bar{f}}\left(s+m, -\frac{1}{N\alpha}\right) y^{-s}\, ds
\end{aligned} \tag{8}$$

for every $M \in \mathbb{Z}_{\geq 0}$.

Proof. This was essentially done in [1, §2]; we reproduce the argument here for the sake of completeness. Let $z = \alpha + iy$, $\beta = -1/N\alpha$ and $u = y/\alpha$. Then

$$-\frac{1}{Nz} = \beta + i|\beta u| - \frac{\beta u^2}{1+iu},$$

so that

$$\epsilon(-i\sqrt{N}z)^{-k}\overline{F}\left(-\frac{1}{Nz}\right) = \epsilon(-i\sqrt{N}\alpha)^{-k} \sum_{n=1}^{\infty} c_{\bar{f}}(n) e(\beta n) e^{-2\pi n|\beta u|}(1+iu)^{-k} e\left(-\frac{n\beta u^2}{1+iu}\right).$$

Next,

$$\begin{aligned}
(1+iu)^{-k} e\left(-\frac{n\beta u^2}{1+iu}\right) &= \sum_{j=0}^{\infty} (-iu)^j (1+iu)^{-j-k} \frac{(-2\pi n|\beta u|)^j}{j!}\\
&= \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} \binom{j+k+\ell-1}{\ell} (-iu)^{j+\ell} \frac{(-2\pi n|\beta u|)^j}{j!}\\
&= \sum_{m=0}^{\infty} (-iu)^m \sum_{j=0}^{m} \binom{m+k-1}{m-j} \frac{(-2\pi n|\beta u|)^j}{j!}.
\end{aligned}$$

Note further that for any $M, K \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned}
&\left|\sum_{m=M}^{\infty} (-iu)^m \sum_{j=0}^{m} \binom{m+k-1}{m-j} \frac{(-2\pi n|\beta u|)^j}{j!}\right|\\
&\qquad \leq (2\pi n|\beta u|)^{-K} K! \sum_{m=M}^{\infty} |u|^m \sum_{j=0}^{m} \binom{m+k-1}{m-j}\binom{j+K}{j} \frac{(2\pi n|\beta u|)^{j+K}}{(j+K)!}\\
&\qquad \leq (\pi n|\beta u|)^{-K} K! (3/2)^{k-1} \sum_{m=M}^{\infty} (3|u|)^m e^{2\pi n|\beta u|}\\
&\qquad \ll_{\alpha,M,K} |u|^{M-K} n^{-K} e^{2\pi n|\beta u|},
\end{aligned}$$

since $|u| \le 1/4$. Hence, substituting the definition of u, we have

$$\epsilon(-i\sqrt{N}z)^{-k}\overline{F}\left(-\frac{1}{Nz}\right) = O_{\alpha,M,K}\Big(y^{M-K}\sum_{n=1}^{\infty}|c_{\bar f}(n)|n^{-K}\Big)$$
$$+\,\epsilon(-i\sqrt{N}\alpha)^{-k}\sum_{m=0}^{M-1}\left(\frac{-iy}{\alpha}\right)^m\sum_{j=0}^{m}\binom{m+k-1}{m-j}\sum_{n=1}^{\infty}c_{\bar f}(n)e(\beta n)\frac{1}{j!}\left(-\frac{2\pi n y}{N\alpha^2}\right)^j e^{-\frac{2\pi n y}{N\alpha^2}}.$$

Choosing $K=\lfloor\frac{k-1}{2}\rfloor+2$, the error term converges and gives the estimate $O_{\alpha,M}(y^{M-K})$.

As for the other terms, we have

$$y^m\sum_{n=1}^{\infty}c_{\bar f}(n)e(\beta n)\frac{1}{j!}\left(-\frac{2\pi n y}{N\alpha^2}\right)^j e^{-\frac{2\pi n y}{N\alpha^2}} = \frac{y^{j+m}}{j!}\frac{d^j}{dy^j}\sum_{n=1}^{\infty}c_{\bar f}(n)e(\beta n)e^{-\frac{2\pi n y}{N\alpha^2}}$$
$$=\frac{y^{j+m}}{j!}\frac{d^j}{dy^j}\frac{1}{2\pi i}\int_{\Re(s)=m+\frac{k}{2}+1}\left(\frac{N\alpha^2}{2\pi}\right)^s\Gamma(s)D_{\bar f}(s,\beta)y^{-s}\,ds$$
$$=\frac{1}{2\pi i}\int_{\Re(s)=\frac{k}{2}+1}\binom{-s-m}{j}\left(\frac{N\alpha^2}{2\pi}\right)^{s+m}\Gamma(s+m)D_{\bar f}(s+m,\beta)y^{-s}\,ds.$$

Moreover, by the Chu–Vandermonde identity we have

$$\sum_{j=0}^{m}\binom{m+k-1}{m-j}\binom{-s-m}{j}=\binom{-s+k-1}{m}=(-1)^m\binom{s+m-k}{m}.$$

Collecting these strands together, we arrive at (8). □

Lemma 6. *For any $\alpha\in\mathbb{Q}^\times$ there are numbers $P_j(\alpha)$, $j=0,1,2,\dots$, such that*

$$B(\alpha+iy)=\sum_{j=0}^{M-1}P_j(\alpha)y^j+O_{\alpha,M}(y^M)$$

for all $M\in\mathbb{Z}_{\ge 0}$ and $y\in\big(0,\frac{|\alpha|}{4}\big]$.

Proof. For $z=\alpha+iy$, we have

$$(9)\qquad (-iz)^{-s}=e^{i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|\alpha|^{-s}\left(1+\frac{iy}{\alpha}\right)^{-s}=e^{i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|\alpha|^{-s}\sum_{j=0}^{\infty}\binom{-s}{j}\left(\frac{iy}{\alpha}\right)^j.$$

Since $y\le\frac{|\alpha|}{4}$, the crude bound

$$\left|\binom{-s}{j}\right|=\left|\binom{s+j-1}{j}\right|\le 2^{|s|+j}$$

yields

$$\sum_{j=M}^{\infty}\binom{-s}{j}\left(\frac{iy}{\alpha}\right)^j\ll_{\alpha,M}2^{|s|}y^M.$$

Hence, if we truncate the sum in (9) at M and substitute it for $(-iz)^{-s}$ in the definition of $B(z)$, then since the contour $\mathcal{C}$ is compact, the first integral of the error term converges to give an $O_{\alpha,M}(y^M)$ error overall. Similarly, by standard estimates, along the line $\Re(s)=k-\frac{1}{2}$

the function $e^{i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|\alpha|^{-s}\Lambda_f(s)$ has at most polynomial growth, and $\frac{\pi^2}{\sin^2(\pi s)} \ll e^{-2\pi|s|}$. Since $e^{2\pi} > 2$, the second integral of the error term converges as well, and the lemma follows with

$$\begin{aligned} P_j(\alpha) &= \frac{1}{2\pi i}\int_{\mathcal{C}} (-i\alpha)^{-j} e^{i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|\alpha|^{-s}\binom{-s}{j}\Delta_f(s)\,ds \\ &\quad + \frac{1}{2\pi i}\int_{\Re(s)=k-\frac{1}{2}} (-i\alpha)^{-j} e^{i\frac{\pi}{2}\operatorname{sgn}(\alpha)s}|\alpha|^{-s}\binom{-s}{j}\Lambda_f(s)\frac{\pi^2}{\sin^2(\pi s)}\,ds. \end{aligned}$$

□

Now, to conclude the proof, let us define

$$\begin{aligned} g(y) &= F(\alpha+iy) + A(\alpha+iy) - \sum_{j=0}^{M-1} P_j(\alpha) y^j \chi_{(0,|\alpha|/4]}(y) \\ &\quad - \epsilon N^{-\frac{k}{2}} \sum_{m=0}^{M-1} \frac{(-i\alpha)^{-m-k}}{2\pi i} \int_{\Re(s)=\frac{k}{2}+1} \binom{s+m-k}{m}\left(\frac{N\alpha^2}{2\pi}\right)^{s+m} \\ &\qquad\qquad \cdot \Gamma(s+m) D_{\bar{f}}\left(s+m, -\frac{1}{N\alpha}\right) y^{-s}\,ds, \end{aligned}$$

where $\chi_{(0,|\alpha|/4]}(y) = 1$ if $y \in \left(0, \frac{|\alpha|}{4}\right]$ and 0 otherwise. Combining Lemmas 3, 5 and 6, we have that $g(y) = O_{\alpha,M}(y^{M-\lfloor\frac{k+3}{2}\rfloor})$ for $y \in \left(0, \frac{|\alpha|}{4}\right]$. On the other hand, it is easy to see that $g(y)$ decays rapidly as $y \to \infty$. Thus, $\frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty g(y) y^{s-1}\,dy$ defines a holomorphic function for $\Re(s) > \lfloor\frac{k+3}{2}\rfloor - M$.

Note that

$$\frac{(2\pi)^s}{\Gamma(s)}\int_0^\infty \sum_{j=0}^{M-1} P_j(\alpha) y^j \chi_{(0,|\alpha|/4]}(y) y^s \frac{dy}{y} = \frac{(2\pi)^s}{\Gamma(s)} \sum_{j=0}^{M-1} P_j(\alpha)\frac{(|\alpha|/4)^{s+j}}{s+j}$$

extends to an entire function of s. Together with (7) and Lemma 4, this shows that (2) is holomorphic for $\Re(s) > \lfloor\frac{k+3}{2}\rfloor - M$. Finally, we replace M by $M+1$ and discard the final term of the sum over m to see that (2) is in fact holomorphic for $\Re(s) > \frac{k+1}{2} - M$. □

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