Spectral Statistics for Chaotic Systems with Discrete Symmetries

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When I started to think about it, I felt the main problem was to explain how the electrons could sneak by all the ions in a metal...

...By straight forward Fourier analysis I found to my delight that the wave differed from the plane wave of free electrons only by a periodic modulation.

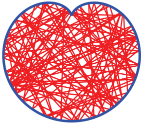
F. Bloch

Contents

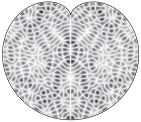
- **Properties of quantum systems with chaos and symmetry**
- **Explaining these features using semiclassical analysis**
- **Further predictions arising from analytical results**

Quantum Chaos

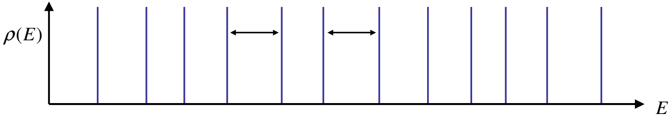
Quantum systems whose classical counterparts are chaotic



$$\rightarrow \quad \hat{H}\psi_n = E_n\psi_n \quad \rightarrow$$



Energy levels tend to 'repel' each other



$$\rho(E) = \sum_n \delta(E - E_n)$$

Quantum Chaos

Energy level distribution given by RMT Bohigas et al. '84

Time-Reversal Symmetry	Example	Hamiltonian	RMT Ensemble
None	Particle in a magnetic field	Hermitian	GUE
$T^2 = 1$	Lasing modes in a chaotic cavity	Hermitian, Real	GOE
$T^2 = -1$	Spin-½ particle	Hermitian, Quarternion-Real	GSE

Correlation function

$$R(\epsilon) = \frac{1}{\bar{\rho}^2} \left\langle \rho\left(E + \frac{\epsilon}{2\pi\bar{\rho}}\right) \rho\left(E - \frac{\epsilon}{2\pi\bar{\rho}}\right) \right\rangle_E - 1$$

Quantum Chaos

Energy level distribution given by RMT Bohigas et al. '84

Time-Reversal Symmetry	**Example**	**Hamiltonian**	**RMT Ensemble**
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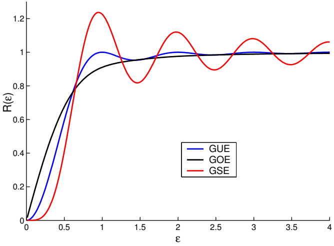
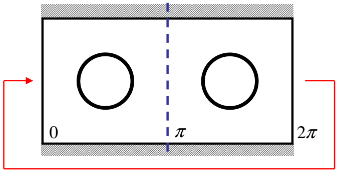
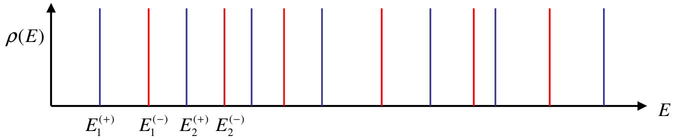


2-fold Symmetry

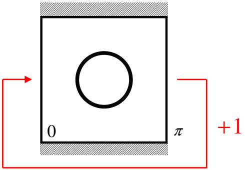


$$\psi_n(x + 2\pi) = \psi_n(x) \qquad \psi_n(x + \pi) = c\psi_n(x) \;\Rightarrow\; c = \pm 1$$

Hilbert space split into $\psi_n^{(+)}$ and $\psi_n^{(-)}$

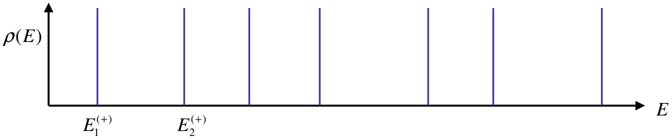


2-fold Symmetry

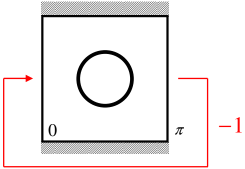


$$\psi_n(x+\pi) = \psi_n(x)$$

$E_n^{(+)}$ are GOE distributed

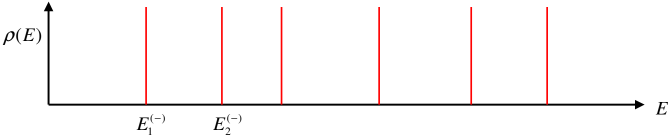


2-fold Symmetry

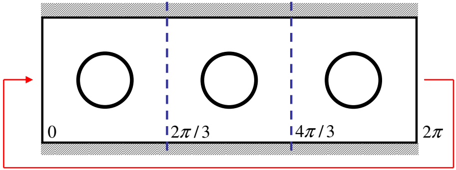


$$\psi_n(x+\pi) = -\psi_n(x)$$

$E_n^{(-)}$ are GOE distributed



3-fold Symmetry



$$\psi_n(x+2\pi) = \psi_n(x), \qquad \psi_n(x+2\pi/3) = c\psi_n(x) \;\;\Rightarrow\;\; c^3 = 1$$

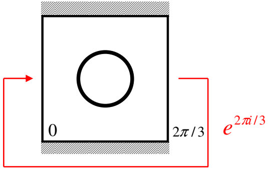
Obtain 3 subspaces, given by

$$\psi_n^{(\alpha)}(x+2\pi/3) = e^{2\pi i\alpha/3}\psi_n^{(\alpha)}(x) \quad \alpha = 0,1,2$$

Where

$$\psi_n^{(0)}(x+2\pi/3) = \psi_n^{(0)}(x)$$

3-fold Symmetry



$$\psi_n^{(1)}(x + 2\pi/3) = e^{2\pi i/3}\psi_n^{(1)}(x)$$

Time-reversal symmetry broken - GUE distributed!!! Leyvraz et al. '96

$$\hat{T}\psi_n^{(1)}(x + 2\pi/3) = \psi_n^{(1)}(x + 2\pi/3)^* = e^{-2\pi i/3}\psi_n^{(1)}(x)^*$$

TR invariance in full system $\Rightarrow$ 2-fold degeneracy

$$\psi_n^{(1)}(x)^* = \psi_n^{(2)}(x) \;\;\Rightarrow\;\; E_n^{(1)} = E_n^{(2)}$$

Semiclassical Analysis

Gutzwiller trace formula (without symmetry)

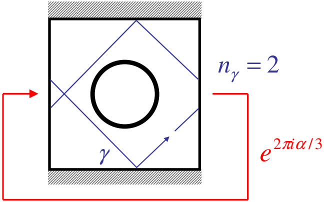
$$\rho(E) = \sum_n \delta(E - E_n) = \bar{\rho} + \mathrm{Re} \sum_\gamma A_\gamma e^{iS_\gamma(E)/\hbar}$$

Connects energy levels E_n to actions S_γ of periodic orbits γ

Correlation Function

$$R(\epsilon) = \mathrm{Re} \frac{1}{2\bar{\rho}^2} \left\langle \sum_{\gamma,\gamma'} A_\gamma A^*_{\gamma'} e^{i(S_\gamma - S_{\gamma'})/\hbar} e^{i(T_\gamma + T_{\gamma'})\epsilon/2T_H} \right\rangle_E$$

Semiclassical Analysis



Amended Gutzwiller Formula Robbins '89

$$\rho_\alpha(E) = \sum_n \delta(E - E_n^{(\alpha)}) = \bar{\rho}_\alpha + \mathrm{Re} \sum_\gamma e^{2\pi i n_\gamma \alpha/3} A_\gamma e^{iS_\gamma(E)/\hbar}$$

Correlation Function

$$R_\alpha(\epsilon) = \mathrm{Re} \frac{1}{2\bar{\rho}^2} \left\langle \sum_{\gamma,\gamma'} e^{2\pi i(n_\gamma - n_{\gamma'})\alpha/3} A_\gamma A^*_{\gamma'} e^{i(S_\gamma - S_{\gamma'})/\hbar} e^{i(T_\gamma)\epsilon/T_H} \right\rangle_E$$

Diagonal Approximation

Main contribution occurs for $S_\gamma = S_{\gamma'}$ Berry '85

$$R_\alpha^{\text{diag}}(\epsilon) = \text{Re}\frac{1}{2\bar{\rho}^2}\left\langle \sum_\gamma \left[e^{2\pi i(n_\gamma - n_\gamma)\alpha/3} + e^{2\pi i(n_\gamma + n_\gamma)\alpha/3}\right] |A_\gamma|^2 e^{i(T_\gamma + T_{\gamma'})\epsilon/2T_H} \right\rangle_E$$

Seperate n_γ from periodic orbit sum Keating, Robbins '96

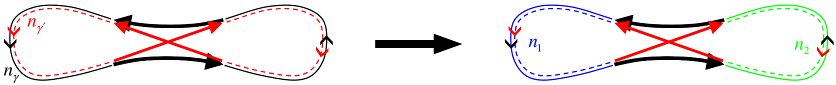
$$R_\alpha^{\text{diag}}(\epsilon) = \left[\frac{1}{3}\sum_{n=0}^{2} 1 + e^{4\pi n i\alpha/3}\right]\left(-\frac{1}{2\epsilon^2}\right)$$

Obtain GOE/GUE diagonal approximation

$$\begin{aligned} R_0(\epsilon) &= \text{Re}\left[-\frac{1}{\epsilon^2} - \frac{1}{i\epsilon^3} + \frac{3}{2\epsilon^4} + \ldots +\right] + \text{Re}\left[\frac{1}{4\epsilon^4} + \frac{2}{i\epsilon^5} + \ldots +\right] e^{2i\epsilon} \\ R_{1/2}(\epsilon) &= \text{Re}\left[-\frac{1}{2\epsilon^2}\right] + \text{Re}\left[\frac{1}{2\epsilon^2}\right] e^{2i\epsilon} \end{aligned}$$

Encounter Pairs

Sub-leading order arises from single 2-encounter Sieber, Richter '01



$$R_\alpha^{\text{sub}}(\epsilon) = \text{Re}\left[\frac{1}{3^2}\sum_{n_\gamma, n_{\gamma'}} e^{2\pi(n_\gamma - n_{\gamma'})i\alpha/3} + e^{2\pi(n_\gamma + n_{\gamma'})i\alpha/3}\right]\left(-\frac{1}{4i\epsilon^3}\right)$$

Winding numbers of γ and γ'

$$n_\gamma = n_1 + n_2 \qquad n_{\gamma'} = n_1 - n_2$$

Change to winding numbers of the links

$$R_\alpha^{\text{sub}}(\epsilon) = \text{Re}\left[\frac{1}{3^2}\sum_{n_1, n_2 = 0}^{2} e^{4\pi n_2 i\alpha/3} + e^{4\pi n_1 i\alpha/3}\right]\left(-\frac{1}{4i\epsilon^3}\right)$$

Results

$$
\begin{aligned}
R_0(\epsilon) &= \mathrm{Re}\left[-\frac{1}{\epsilon^2} - \frac{1}{i\epsilon^3} + \frac{3}{2\epsilon^4} + \ldots +\right] + \mathrm{Re}\left[\frac{1}{4\epsilon^4} + \frac{2}{i\epsilon^5} + \ldots +\right] e^{2i\epsilon} \\
R_{1/2}(\epsilon) &= \mathrm{Re}\left[-\frac{1}{2\epsilon^2}\right] + \mathrm{Re}\left[\frac{1}{2\epsilon^2}\right] e^{2i\epsilon}
\end{aligned}
$$

Pseudo-Real Symmetries

Example: Quaternion group Q_8 - subgroup of $SU(2)$

$$Q_8 = \{\pm 1, \pm I, \pm J, \pm K\}$$

Where

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

Amended Gutzwiller trace formula

$$\rho_\alpha(E) = \sum_n \delta(E - E_n^{(\alpha)}) = \bar{\rho}_\alpha + \mathrm{Re} \sum_\gamma \mathrm{Tr} M_\alpha(g) A_\gamma e^{iS_\gamma(E)/\hbar}$$

Result

$$\begin{aligned} R_\alpha(\epsilon) &= \mathrm{Re}\left[-\frac{1}{\epsilon^2} + \sum_{k=3}^{\infty} \frac{(k-3)!(k-1)}{2i^k} \frac{(-1)^k}{\epsilon^k}\right] + \mathrm{Re}\left[\sum_{k=3}^{\infty} \frac{(k-3)!(k-3)}{2i^k} \frac{1}{\epsilon^k}\right] e^{2i\epsilon} \\ R_{\mathrm{GOE}}(\epsilon) &= \mathrm{Re}\left[-\frac{1}{\epsilon^2} + \sum_{k=3}^{\infty} \frac{(k-3)!(k-1)}{2i^k} \frac{1}{\epsilon^k}\right] + \mathrm{Re}\left[\sum_{k=3}^{\infty} \frac{(k-3)!(k-3)}{2i^k} \frac{1}{\epsilon^k}\right] e^{2i\epsilon} \end{aligned}$$

Pseudo-Real Symmetries

By accounting for an additional 2-fold degeneracy we obtain

$$R_\alpha(\epsilon) = \mathrm{Re}\left(-\frac{1}{(2\epsilon)^2} + \sum_{k=3}^{\infty} \frac{(k-1)(k-3)!}{2i^k} \frac{(-1)^k}{(2\epsilon)^k} + e^{2i\epsilon} \frac{\pi}{2} \frac{(i+2\epsilon)}{(2\epsilon)^2} + \sum_{k=3}^{\infty} \frac{(k-3)(k-3)!}{2i^k} \frac{e^{2i(2\epsilon)}}{(2\epsilon)^k}\right)$$

Pseudo-Real Symmetries

By accounting for an additional 2-fold degeneracy we obtain

$$R_\alpha(\epsilon) = R_{\mathsf{GSE}}(\epsilon), \qquad \alpha \text{ pseudo-real}$$

Pseudo-Real Symmetries

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How to obtain GSE statistics

1. Hamiltonian H quaternion real

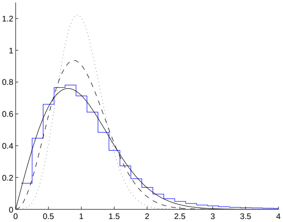
   $\hat{T}^2 = -1$: Require half-integer spin!

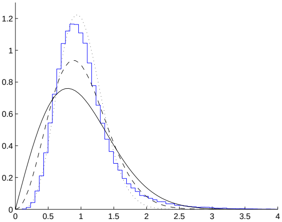
2. Hamiltonian H real and *invariant under PR symmetry group*

   Can use zero-spin particles, provided system has right symmetry

Applications to RMT

Comparison of Hamiltonians with and without symmetry





- H_1, real, Hermitian, Gaussian-distributed elements
- H_2, real, Hermitian, symmetric under Q_8-symmetry group

$$H_2 = \frac{1}{|G|} \sum_g T^{-1}(g) H_1 T(g)$$

Where $T(g)$ are orthogonal

Conclusions and Outlook