Applied Dynamical Systems

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2 Dynamics and time

2.1 Maps and flows

We have seen two types of dynamical systems so far, differential equations such as Newton's or Lorenz's equations, and discrete time systems, such as the logistic map. In each case there is a on-parameter family of maps $\Phi^t : X \to X$ on the space X (say $\mathbb{R}^d$) which moves forward the dynamics in time by an amount t for any initial time s:[1]

$$\mathbf{x}(s+t) = \Phi^t(\mathbf{x}(s))$$

In the map case $t \in \mathbb{N}$ and $\Phi^1 = \Phi$ is just the original map, while for differential equations, $(d/dt)\mathbf{x} = f(\mathbf{x})$, $t \in \mathbb{R}$ and the Φ^t is a called a flow, found (if possible) by solving the differential equation for arbitrary initial condition $\mathbf{x}$:

$$\frac{d}{dt}\Phi^t(\mathbf{x}) = f \circ \Phi^t(\mathbf{x}), \qquad \Phi^0(\mathbf{x}) = \mathbf{x}$$

Example 2.1. *For the harmonic oscillator* $(d/dt)(x, v) = (v, -x)$ *we have* $\Phi^t(x, v) = (x\cos(t) + v\sin(t), -x\sin(t) + v\cos(t))$.

We can easily see that for either a map or flow, Φ satisfies the semigroup property:

$$\Phi^s \circ \Phi^t = \Phi^{s+t}$$

and hence

$$\Phi^0(\mathbf{x}) = \mathbf{x}$$

Saying a map is **invertible** means that $\Phi^{-1} : X \to X$ is uniquely defined, so the Φ^t form a group under composition.[2] Normally, flows are invertible from the Picard-Lindelöf theorem.[3] For either invertible or non-invertible maps, Φ^{-1} is a function on sets $A \subset X$

$$\Phi^{-1}(A) = \{x \in X : \Phi(x) \in A\}$$

[1] We assume here that the system is autonomous; more generally we could consider Φ a function of both s and t. Generalising dynamical results to non-autonomous systems is a popular source of research problems. In some cases it is useful to add the time as an extra variable.

[2] It is also possible to use a semi-group larger than $\mathbb{N}$ or $\mathbb{R}$ as a "time" variable; this comes under the name of "group actions," also a popular research topic. A famous example is Furstenberg's conjecture that the system $\Phi^{(i,j)}(x) = 2^i 3^j x \pmod 1$ has no nontrivial invariant measures. H. Furstenberg Mathematical Systems Theory **1** 1-49 (1967).

[3] Piecewise smooth flows, which need not be invertible, form another rich source of research problems. See for example M. di Bernado and S. J. Hogan, Trans. Roy. Soc. A **368**, 4915-4935 (2010).

giving the **pre-images** of a point or larger set.[4] Invertible systems may also be **(time)-reversible**; this means that there is a transformation $i : X \to X$ satisfying

$$i \circ \Phi^t \circ i = \Phi^{-t}$$

Setting $t = 0$ we note that i^2 is the identity, that is, i is an involution (hence the notation). For example, Newton's law of gravitation is reversible, with i reversing all the velocities.[5]

Often a system can have more than one description, using either a map or a flow. These can be related as follows: A **time-one** or **stroboscopic** map is obtained by considering a flow Φ^t and treating Φ^1 (or more generally some $\Phi^{\Delta t}$) as a map in its own right. An alternative, and probably more useful approach is the **Poincaré** map, defining a hypersurface Y and stopping whenever this is reached. If the time from one event to the next is

$$\tau(\mathbf{y}) := \min_{t>0}\{t : \Phi^t(y) \in Y\}$$

the Poincaré map can be defined as

$$F(\mathbf{y}) = \Phi^{\tau(\mathbf{y})}(\mathbf{y})$$

The reverse process is called a **suspension**: Given a map $F : Y \to Y$ and "roof function" $\tau : Y \to (0, \infty)$, we can construct a flow on the space $X = \{(\mathbf{y}, s) : \mathbf{y} \in Y, 0 \leq s < \tau(\mathbf{y})\}$ in the natural way - increase s until $\tau(\mathbf{y})$ is reached, then apply F and set $s = 0$.

Example 2.2. *For the harmonic oscillator above, and the Poincaré section (hypersurface) $x = 1$ the Poincaré map is $F(y) = -y$ (taking $y = v$), but note that not all trajectories of the original flow reach it.*[6] *The roof function is*

$$\tau(y) = \begin{cases} 2\arctan|y| & y > 0 \\ 2\pi & y = 0 \\ 2\pi - 2\arctan|y| & y < 0 \end{cases}$$

A **dynamical billiard** consists of a point particle that moves freely except for mirror-like reflections with the boundary, that is, angle of reflection equals angle of incidence. The flow is discontinuous (in the momentum variable) at collisions, so it is natural to consider a Poincaré section consisting of the boundary.

An **induced map** is obtained starting from a map $\Phi : X \to X$ and a subset Y of full dimension, and proceeding in the same manner. This can help if (almost) all orbits pass through Y and the induced dynamics has more uniform properties.

[4] Sometimes set-valued dynamics is considered in the forward time direction, too.

[5] More on dynamical reversibility can be found in J. A. G. Roberts and G. R. W. Quispel, Phys. Rep. **216**, 63-177 (1992).

[6] If you start on the Poincaré section, there are often results that show you almost certainly return, for example the Poincaré recurrence theorem discussed in chapter 6.

Example 2.3. *The Farey map is*

$$\Phi(x) = \begin{cases} \frac{x}{1-x} & x < 1/2 \\ \frac{1-x}{x} & x > 1/2 \end{cases}$$

which has very slow behaviour near $x = 0$. Iterating the left branch we find for small x

$$\Phi_L^2(x) = \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{x}{1-2x}$$

$$\Phi_L^3(x) = \frac{x}{1-3x}$$

$$\Phi_L^n(x) = \frac{x}{1-nx}$$

Suppose we induce on the right branch, ie $Y = [1/2, 1)$. We take τ iterations to return to the right branch, ie

$$F(y) = \Phi_L^{\tau-1}(\Phi_R(y)) = \frac{\frac{1-y}{y}}{1 - (\tau-1)\frac{1-y}{y}} = \frac{1-y}{\tau y - (\tau - 1)}$$

Equivalently since we have alternating strings of $\tau - 1$ iterations of Φ_L followed by a single Φ_R we can consider the dynamics immediately after the right branch iteration, ie

$$G(x) = \Phi_R(\Phi_L^{\tau-1}(x)) = \frac{1 - \frac{x}{1-(\tau-1)x}}{\frac{x}{1-(\tau-1)x}} = \frac{1}{x} - \tau$$

This the famous Gauss map, and the τ values give the continued fraction expansion of x. It is much faster to calculate, and no longer has a region near zero that behaves very differently to the rest of the map, however we have replaced a map with two branches by one with an infinite number.

2.2 Numerical considerations

If we have an explicit equation for a map, it is straightforward to simulate it numerically, although there are issues to do with instability and finite precision that we will discuss later.

Similarly, a flow (ordinary differential equation) can be simulated using standard numerical techniques, which effectively approximate the stroboscopic map using small step size, for example the simplest (Euler) method for approximating $\dot{\mathbf{x}} = f(\mathbf{x})$ is

$$x_{t+h} = x_t + hf(x_t)$$

This requires extremely small step size for accuracy (which then takes longer, and suffers from round-off error); a slightly better algorithm is the midpoint method, which (approximately) calculates the derivative from the midpoint of the interval

$$x_{t+h} = x_t + hf(x_t + \frac{h}{2}f(x_t))$$

There are a variety of more accurate and reliable methods discussed in texts on numerical analysis and used by numerical software.[7]

In order to compute a Poincaré map, we need to do more work, however. We need to simultaneously solve the differential equation, and an algebraic equation in a single variable, the time. Fast methods for algebraic equations $g(t) = 0$ include Newton's method

$$t_{n+1} = t_n - \frac{g(t_n)}{g'(t_n)}$$

— another source of discrete dynamical systems. This converges quadratically (double the number of digits at each step) for good initial guesses, but no guarantee of convergence otherwise.[8] If there is a change of sign, $g(t_1)g(t_2) < 0$, the bisection method is guaranteed to find a solution, with linear convergence. There may however be several changes of sign in the initial interval. So, ideally there should also be a rigorous lower bound on the time.

Example 2.4. *Find the smallest positive solution of $at = \sin t$ for $0 < a < 1$. Choose t_0 to be a small positive value. Since $g(t) = \sin(t) - at$, we know $g''(t) \geq -1$ and so, integrating twice*

$$g(t) \geq -(t - t_n)^2/2 + (t - t_n)g'(t_n) + g(t_n)$$

Thus the desired solution of $g(t) = 0$ is greater than the smallest root (greater than t_n) of this quadratic equation. Also, since we match both the value of $g(t_n)$ and its derivative, the iteration will be quadratically convergent like the Newton method.

An alternative approach is a simple change of variable. Suppose we have a system with d variables, and we can write the system in the form

$$\frac{dg}{dt} = f_1(g, x_2, \ldots, x_d)$$

$$\frac{dx_j}{dt} = f_j(g, x_2, \ldots, x_d), \qquad j \geq 2$$

where the Poincaré section is at $g = 0$. Then, making g the independent variable we have

$$\frac{dt}{dg} = \frac{1}{f_1}$$

$$\frac{dx_j}{dg} = \frac{dx_j}{dt}\frac{dt}{dg} = \frac{f_j}{f_1}$$

In this form we can integrate directly to $g = 0$.

[7] Accuracy is improved by combining linear combinations of the function evaluated at different times, the Runge-Kutta methods. Stability is improved by including x_{t+h} on the RHS, requiring Newton's method or similar at each step, the implicit methods. Often these approaches are combined.

[8] Intriguingly, multiplying the last term by a random variable may sometimes be guaranteed to converge almost surely: H. Sumi arxiv:1608.05230.

