Applied Dynamical Systems Problem Sheet 1

1. Hamilton's equations (given in section 7 of the notes) for a system with Hamilonian function $H(\mathbf{q}, \mathbf{p})$ are

$$\frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}}$$

   Show that for any Hamiltonian dynamics, the Hamiltonian function itself is conserved; this function is normally interpreted as the total energy. For a pendulum with $H(q,p) = p^2/(2m) - mgl\cos q$ (here, q is the angle measured from the lowest point and the constants are mass m, gravity g and length l), write down the equations of motion and show explicitly that the energy is conserved.

2. Find the general solution to $\dot{x} = \sqrt{x}$ [Harder than it looks!]. Does this equation satisfy the Picard-Lindelöf theorem?

3. Investigate the unpredictability of weather: Each day at the same time for several days or weeks, record the predicted top temperatures in your home town given on the BBC (or similar site) on a spreadsheet. Find the variance of the errors for temperatures predicted n days in advance for $n = 1, 2, 3, \ldots$.

4. Investigate a transition to chaos experimentally. By carefully varying the flow of water through a slowly drippng tap, identify as many as possible regular and chaotic regimes. Can you observe a period doubling cascade?

5. Find all fixed points and period 2 points of the logistic map analytically.

6. For a damped oscillator $\dot{x} = v, \quad \dot{v} = -x - \alpha v$, $(\alpha > 0)$, determine the flow, the time-one map, and the map corresponding to the Poincarè section $x = 0$. Is this flow invertible? Reversible?

7. Now consider the damped oscillator with a Poincaré section $x = 1$. Given $\alpha = 0.01$, $x(0) = 2$, $v(0) = 0$, find the times at which $x = 1$ by solving the ODE and Poincaré surface condition numerically using a computer language of your choice.

