Applied Dynamical Systems Solution Sheet 2

1. (a) $rx(1-x)=0$, $Df=r(1-2x)$. So, the fixed points are at $x=0$, which is unstable for $r>0$ and stable for $r<0$; $x=1$ which is unstable for $r<0$ and stable for $r>0$. At $r=0$ all points are fixed; since perturbations do not grow, this means the fixed points are stable but not asymptotically stable.

   (b) We can write this as

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= b(1-x^2)y - x\end{aligned}$$

   The fixed point condition gives $y=0$ and hence $x=0$ also. We have

$$Df = \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}$$

   with eigenvalues $\lambda = (b/2) \pm \sqrt{(b/2)^2 - 1}$. Thus for $-2<b<0$ we have two complex values with negative real part (stable focus), $0<b<2$ gives positive real part (unstable focus) and $|b|>2$ two real values (unstable/stable node for positive/negative respectively). The marginal cases are $b=0$, $\lambda=\pm i$ (centre; just a harmonic oscillator) and $b=\pm 2$, a single eigenvalue $\lambda=\pm 1$, but because this is not a multiple of the unit matrix it is an unstable/stable degenerate node for positive/negative respectively.

   (c) For positive values of the parameters we find from the first equation $y=x$, then from the third equation $z=x^2/\beta$ then the second equation becomes

$$-x^3/\beta + \rho x - x = 0$$

   so $x=0$ or $x=\pm\sqrt{\beta(\rho-1)}$, assuming $\rho>1$. We have

$$Df = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

   Thus the fixed point at the origin $(0,0,0)$ has eigenvalues $-\beta$, $-(1+\sigma)/2 \pm \sqrt{(\sigma+1)^2/4 + \sigma(\rho-1)}$. If $\rho<1$ these are all stable (negative real part), so we have a node, while for $\rho>1$ one becomes positive and we have a saddle with one unstable and two stable eigenvalues.

The other fixed points both lead to the characteristic equation for the eigenvalues

$$\lambda^3 + (1 + \beta + \sigma)\lambda^2 + \beta(\rho + \sigma)\lambda + 2\sigma(\rho - 1) = 0$$

All coefficients are positive, so there cannot be any unstable real eigenvalues. If there are complex eigenvalues we can write them in the form $A, B \pm iC$. It is straightforward to express the coefficients of a general cubic $x^3 + ax^2 + bx + c$ in terms of A, B and C, and hence show that $c - ab = 2B((A+B)^2 + C^2)$, thence that the sign of this expression is the same as that of B. We thus find that if there are complex conjugate eigenvalues they will be positive if

$$2\sigma\beta(\rho - 1) + (1 + \beta + \sigma)\beta(\rho - \sigma) > 0$$

That is,

$$\rho > \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1}$$

Determining whether the roots are all real or there is a complex conjugate pair is straightforward using the discriminant of the cubic, easiest with symbolic algebra packages. Thus depending on the parameters we have a stable node, or a stable or unstable focus with the third direction stable.

2. The period may be found by waiting until the trajectory has reached the limit cycle, then using a Poincare section method (as in the question on sheet 1) to determine the period T (eg time it takes to return to $x = 0$ twice). Then integrate both the original and linearised equations for this time to obtain the stability matrix $D\Phi^t$. The result is $T = 6.66329$ with eigenvalues 1 (corresponding to the flow direction) and 8.59695×10^{-4} which is of magnitude less than one, so indicating stability.

3. (a) $\Phi(x) = \tanh x$ has a fixed point at zero with unit derivative; all points (not just in the neighbourhood of the fixed point) approach the origin asymptotically. (b)

$$\Phi \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

is a linear map with a degenerate spectrum and non-trivial Jordan normal form. We see z is constant; if $z \neq 0$ it makes y grow linearly, which in turn makes x grow quadratically.

4. Following the instructions, we write (with $f(x) = rx(1-x)$ and its linearisation about zero rx so that $\Psi^t(x) = e^{rt}x$)

$$h \circ \Phi^t = \Psi^t \circ h$$

$$(h' \circ \Phi^t)(f \circ \Phi^t) = re^{rt}h$$

At $t = 0$ this becomes

$$h'(x)rx(1-x) = rh(x)$$

Expanding $h(x) = h_1 x + h_2 x^2 + \ldots$, equating terms and summing a geometric series; or alternatively solving by separation of variables, we find a solution

$$h(x) = \frac{ax}{1-x}$$

for arbitrary a. We can choose any conjugation, so set $a = 1$. Then $h^{-1}(x) = x/(1+x)$. Thus we find

$$\Phi^t(x) = h^{-1}(\Psi^t(h(x))) = \frac{e^{rt}x}{1 - x + e^{rt}(x)}$$

which it can be confirmed satisfies the required conditions

$$\Phi^0(x) = x, \qquad \frac{d}{dt}\Phi^t(x) = f(\Phi^t(x))$$

5. We have

$$\begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix} = A \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

This has eigenvalues $(1 \pm \sqrt{5})/2 = \{g, -g^{-1}\}$ and eigenvectors $(1, g)$, $(-g, 1)$ where $g = (1 + \sqrt{5})/2$ is the golden ratio. We construct the transformation matrix C, so that $B = C^{-1}AC$ is the diagonal matrix with entries $\{g, -g^{-1}\}$ using the eigenvectors as columns:

$$C = \begin{pmatrix} 1 & -g \\ g & 1 \end{pmatrix}$$

so that

$$C^{-1} = \frac{1}{g\sqrt{5}} \begin{pmatrix} 1 & g \\ -g & 1 \end{pmatrix}$$

since $g^2 + 1 = g\sqrt{5}$. Thus

$$\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = A^n \begin{pmatrix} F_0 \\ F_1 \end{pmatrix} = C^{-1}B^nC \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} g^n - (-g)^{-n} \\ g^{n+1} - (-g)^{-n-1} \end{pmatrix}$$

from which we can read the formula from the top line.

6. **(a)** Any polynomial, or e^{cz} with $|c| < 1$.

   (b) e^{cz} with $c > 1$.

   (c) $e^{\omega z}$ with ω a complex cube root of unity. If you want a real function, $e^{\omega z} + e^{\omega^2 z}$.

   (d) $e^{i\pi cz}$ with $c \notin \mathbb{Q}$.

   (e)

$$\sum_{n=0}^{\infty} \frac{a_n}{n!}$$

   with a_n a typical realisation of an iid random variable with support the whole of $\mathbb{C}$ and density decaying polynomially at infinity.

Linear operators on infinite dimensional spaces have much richer behaviour than the finite case, for which a dense orbit is not possible.

