Applied Dynamical Systems Solution Sheet 3

1. There is always a fixed point at $y = 1$. If we write $f(y) = a \ln y + y - 1$, we find $f'(1) = a + 1$ so the fixed point is marginal only for $a = -1$. When $a < 0$, $f(y)$ tends to infinity for both $y \to 0$ and $y \to \infty$, so there is a second fixed point y^* with $f'(y^*) < 0$ and $0 < y^* < 1$ for $-1 < a < 0$; and $y^* > 1$ and $f'(y^*) > 0$ for $a < -1$. The two fixed points coincide for $a = -1$, corresponding to a transcritical bifurcation. This is confirmed by the Taylor expansion

$$\frac{d}{dt}(y-1) = (a+1)(y-1) + \frac{1}{2}(y-1)^2 + \dots$$

2. The analysis is very similar to the logistic map.

   Fixed points are $x = 0$ and $x^*_\pm = \pm\sqrt{1 - r^{-1}}$. $f'(0) = r$ and $f'(x^*_\pm) = 3 - 2r$. Thus the fixed points $x^*_\pm$ appear at $r = 1$ in a pitchfork bifurcation, and become unstable at $r = 2$ at period doubling bifurcations. An example of a fold bifurcation is the creation of the period 3 stable/unstable pair at $r \approx 2.45$. At $r = 3\sqrt{3}/2$ the local maximum at $x = 1/\sqrt{3}$ maps to 1 and then 0; beyond this the two attractors at positive and negative x merge in an attractor merging crisis. Finally at $r = 3$ there is a fixed point at the boundary $x = 2/\sqrt{3}$; this is a boundary crisis as for larger r almost all initial conditions are unbounded.

   At $r = 1$ we have for small perturbations $\delta = x$:

$$\delta_{n+1} = \delta_n - \delta_n^3$$

   which (replacing $\delta_{n+1} - \delta_n$ by a derivative) leads to

$$\delta_n = \frac{1}{\sqrt{2n}} + O(n^{-3/2})$$

   At $r = 2$ we have for small perturbations $\delta = x - 1/\sqrt{2}$:

$$\delta_{n+2} = \delta_n - 32\delta_n^3 + O(\delta_n^4)$$

   This leads to

$$\delta_n = \frac{(-1)^n}{\sqrt{32n}} + O(n^{-1})$$

The fold case (creation of period three) is similar but more involved; the normal form gives

$$\delta_{n+3} = \delta_n - c\delta_n^2 + O(\delta_n^3)$$

for some constant c arising from the Taylor expansion of the three times composed map. Assuming $\delta_n > 0$ we find

$$\delta_n = \frac{3}{cn} + O(n^{-2})$$

The Schwarzian derivative is $-6(1+6x^2)/(1-3x^2)^2$ (independent of r) which is clearly negative, and the critical points are quadratic, so the general theory applies; there are 3 monotonic intervals so at most 4 coexisting stable fixed points (in practice either the origin or the pair $x_\pm^*$). Yes, the conditions of a period doubling cascade with the Feigenbaum constants are met.

3. The Jacobian has eigenvalues 0 and 1, which are neutral and unstable respectively. The centre manifold corresponding to the neutral direction is easy to identify: The entire line $x = y$ consists of fixed points. The unstable manifold may be found by solving

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = x + y$$

This is a linear first order equation, with solution

$$y = Ce^x - x - 1$$

for some constant C, equal to 1 for an orbit approaching the origin. Thus it intersects another fixed point when $x = e^x - x - 1$, that is, approximately $x = y = 1.25643$.

4. The centre at $x = 0$ becomes first a stable spiral (underdamped) then a stable node (overdamped) with the addition of damping. The hyperbolic point at $x = \pi$ remains so. Almost all orbits now limit to $x = 0$, thus the stable manifold of this point is now the whole phase space except for the hyperbolic points and their stable manifolds (which now extend outwards to higher $|v|$). The unstable manifold of the fixed point at $x = \pi$ now limits to the point at $x = 0$ rather than to itself.

