Applied Dynamical Systems Solution Sheet 4

1. The plot is displayed on the unit home page. The best numerical method is inverse iteration, with a range of initial conditions.

2. (a) Irreducible but not aperiodic: $A \to B$, $B \to A$ allows each state to reach the other but has period 2. The corresponding dynamical system is ergodic but not weak mixing.

   (b) Aperiodic but not reducible: $A \to A$, $A \to B$, $B \to B$. The corresponding dynamical system is not ergodic.

   (c) Has a greater topological entropy than when restricted to its essential symbols: Full 3-shift on $\{A, B, C\}$, full 2-shift on $\{D, E\}$, connection $C \to D$. The essential symbols are D and E. Again not ergodic.

3. Denote the partition element $[0, 1/3)$ by 0 and the other by 1. All transitions are possible, so the topological entropy is that of the binary shift, namely $\ln 2$. It is clear that each partition element has probability 1/3 to reach 0 and 2/3 to reach 1. Thus after one (or more) iterations, any initial density that is constant on the partition elements is made uniform, $\rho_i = 1$. For a 1D map, the Lyapunov exponent is the time average of $\ln f'$, and since this map is ergodic, this may be replaced by an average over the (uniform) invariant measure. Thus we have

$$\lambda = \frac{1}{3}\ln 3 + \frac{2}{3}\ln\frac{3}{2} = \ln 3 - \frac{2}{3}\ln 2$$

   The KS entropy is equal to this (as follows from the formula for entropy of Markov chains).

4. Denote the partition element $[0, 1/2)$ by 0 and $[1/2, 3/4)$ by 1. The allowed transitions are $0 \to 0$, $0 \to 1$, $1 \to 0$, ie the same as the golden mean beta shift, which has topological entropy $\ln g$. The uniform measure $\rho = 1$ is easily seen to be conditionally invariant with escape rate $-\ln(3/4)$. The non-escaping set is a fractal, containing a piece in each of the two partition elements. The piece in 0 scales to the whole set in one iteration of the map (factor of two in size), while the piece in 1 scales only to the set in 0. Thus using the similarity dimension formula

$$(1/2)^D + (1/4)^D = 1$$

Noting that this is a quadratic equation in $(1/2)^D$ we find that the dimension is $\ln g/\ln 2$.

5. Invariance of the density can be shown using the transfer operator, but it is easier to make use of the conjugation with the tent map. In particular we have $x = \sin^2(\pi y/2)$ in terms of which $\rho(x)dx = \tilde{\rho}(y)dy$ and $\tilde{\rho}$ is constant.

   Similarly for the second part, the eigenvalue equation for the tent map reads
$$\lambda\tilde{\rho}(y) = \frac{1}{2}\left[\tilde{\rho}(y/2) + \tilde{\rho}(1-y/2)\right]$$
   This is satisfied by polynomials; the next simplest, obtained by solving for arbitrary coefficients, is
$$\lambda = \frac{1}{4}, \qquad \tilde{\rho}(y) = 3y^2 - 6y + 2$$
   leading to
$$\frac{1}{\pi\sqrt{x(1-x)}}\left[\frac{12}{\pi^2}\arcsin^2\sqrt{x} - \frac{12}{\pi}\arcsin\sqrt{x} + 2\right]$$

6. The code and its output are given in a separate file.

