Hand in at lecture on Tuesday 20th March: Q. 1, 2.

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1. In lectures we showed how F.T.'s and convolutions can be used to give the solution for the diffusion equation for a general initial condition $u(x,0) = f(x)$ in the form $u(x,t) = \dfrac{1}{\sqrt{4\pi Dt}} \displaystyle\int_{-\infty}^{\infty} f(\xi) e^{-(x-\xi)^2/4Dt}\, d\xi$.

(a) Find an explicit form for the solution when $f(x) = \begin{cases} 1, & 0 < x < a \\ 0, & \text{all other values of } x \end{cases}$

in terms of the error function, $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-s^2}\, ds$.

(b) Using properties of the error function (see lectures), use your solution to part (a) to show that as $a \to \infty$, $u(x,t) \to \frac{1}{2}(1 + \operatorname{erf}(x/\sqrt{4Dt}))$.

2. In this question you will solve the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$ using Fourier transforms.

(a) As usual, let $\widetilde{u}(k,t) = \displaystyle\int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$.

Take the Fourier transform of the PDE and show that $\dfrac{d^2\widetilde{u}}{dt^2} = -k^2c^2\widetilde{u}$.

(b) Write down a general solution of the ODE for $\widetilde{u}$.

(c) Consider the particular initial conditions in which $u(x,0) = 0$, $u_t(x,0) = \delta(x)$. (This is a model of how a piano string vibrates when it is struck, ignoring the effects of the ends.) In this case, find the unique solution of $\widetilde{u}$.

(d*) Invert the transform, by first considering the transform of the function $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and the inverse Fourier transform theorem. Hence show that the solution is given by

$$u(x,t) = \begin{cases} 1/(2c), & |x| < ct \\ 0, & |x| > ct \end{cases}$$

(e) Sketch the solution for times $t = 1/c$, $t = 2/c$. Explain in words what the solution represents.

3*. (a) Let u be a solution of the first-order PDE $u_t + cu_x = 0$ where c is a constant.

Let $\widetilde{u}(k,t) = \displaystyle\int_{-\infty}^{\infty} u(x,t) e^{-ikx}\, dx$ be the F.T. of $u(x,t)$.

Use the PDE for u to show that $\widetilde{u}_t = -ikc\widetilde{u}$.

(b) The initial condition is $u(x,0) = f(x)$ for some given function f. Deduce that $\widetilde{u}(k,t) = f(k)e^{-ikct}$ where $f(k) = \mathscr{F}\{f\}$.

(c) Take inverse transforms to determine the solution $u(x,t) = f(x - ct)$.

[Hint: *Use the convolution theorem plus the fact from §5.3.3 of the notes that* $\mathscr{F}\{\delta(x+\alpha)\} = e^{ik\alpha}$.]

(d) Given the initial data $f(x) = (1 + x^2)^{-1}$, sketch graphs of u as a function of x for $t = 0, 1/c, 2/c$.

4*. This question concerns the one-dimensional wave equation $u_{tt} = c^2 u_{xx}$.

(a) Follow question 3 until part (b), where a general solution is derived, but express your general solution as the sum of two complex exponentials rather than trigonometric functions.

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(b) Now calculate the Fourier inverse, $u(x,t)$ of $\widetilde{u}$, in terms of two functions f, g, whose transforms are the "constants of integration" in the general solution from part (a).

[Hint: *Use convolution and the fact that* $e^{ik\alpha}$ *is the F.T. of* $\delta(x+\alpha)$]

Hence obtain the general solution of the wave equation, considered in Q. 4(b) of Problems 2.

5*. Consider the PDE $u_t = u_{xx} + \alpha u_x$, where α is a positive constant.

(a) Let $\widetilde{u}$ be the F.T. of u with respect to x. Take the FT of the PDE, and show that $\widetilde{u} = Ae^{\beta t}$, where β is a function of k which you should find, and A is an arbitrary function of k.

(b) If u satisfies the initial condition $u(x,0) = \phi(x)$ for all x, find A, and hence find the solution u.

[HINT: Note the result $\mathscr{F}\{f(x+a)\} = e^{ika}\mathscr{F}\{f\}$.]

(c) Write down the fundamental solution satisfying $u(x,0) = \delta(x-a)$ where a is a given number.

6*. This question uses Fourier Transforms to consider the steady-state heat equation in a two-dimensional semi-infinite slab of heat-conducting material occupying $y > 0$, $-\infty < x < \infty$. The governing equation is Laplace's equation, $u_{xx} + u_{yy} = 0$.

Along $y = 0$, the temperature is given as $u(x,0) = f(x)$ and as $y \to \infty$, the temperature must tend to zero, or $u(x,y) \to 0$, as $y \to \infty$.

(a) Take F.T.'s in x of the PDE, and explain why the solution for $\widetilde{u}$ must be of the form $\widetilde{u}(k,y) = C(k)e^{-|k|y}$

(b) Determine $C(k)$ in terms of the F.T. of $f(x)$. Using the fact (proved on Sheet 6, Q.4a) that the F.T. of the function $a/(\pi(a^2+x^2))$ is $e^{-|k|a}$ determine the solution for $u(x,t)$ using convolution.

(c) If $f(x) = \delta(x)$, representing a situation when the temperature is held fixed at zero along the whole of $y = 0$, apart from at $x = 0$ where there is a unit source of heat, show that $u(x,t) = \pi^{-1}y/(y^2+x^2)$.

7*. (a) Use Fourier transforms to solve $u_t = u_{xx} - \alpha u$ with $u(x,0) = \phi(x)$ for all x.

(b) Show that your solution is $e^{-\alpha t}\times$ the solution of $u_t = u_{xx}$ with $u(x,0) = \phi(x)$.

(c) Solve $u_t = u_{xx} - \alpha u$ for $x > 0$ with $u(0,t) = 0$ and $u(x,0) = \phi(x)$ for $x > 0$.

8.** Consider the heat equation on $x > 0$, $u_t = Du_{xx}$ with a BC $u_x(0,t) = -D^{-1}\sin\omega t$ and $u \to 0$ as $x \to \infty$ (this mimics the periodic heating and cooling at the end of a long rod supplied by an external heat flux of $q = \sin\omega t$).

(a) Consider $v_t = Dv_{xx}$ with $v_x(0,t) = e^{i\omega t}$ and $v(x,t) \to 0$ as $x \to \infty$. What is the relationship between u and v ?

(b) Show, by substitution that, $v = Ce^{i\omega t}e^{-\sqrt{i\omega/D}x}$ is a solution of the PDE and BCs and find C. Hence write down a solution to the problem for u.

(c) Use your solution to show that the temperature at the end of the rod lags one eighth of a period behind the flux. [1]

[1] This explains why the hottest time of the day is the afternoon (and not midday, when the sun is highest) and why the hottest time of the year is August (and not June when the sun is highest in the sky).

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