Spherical billiards with almost complete escape

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Abstract

A dynamical billiard consists of a point particle moving uniformly except for mirror-like collisions with the boundary. Recent work has described the escape of the particle through a hole in the boundary of a circular or spherical billiard, making connections with the Riemann Hypothesis. Unlike the circular case, the sphere with a single hole leads to a non-zero probability of never escaping. Here we study variants in which almost all initial conditions escape, with multiple small holes or a thin strip. We show that equal spacing of holes around the equator is an efficient means of ensuring almost complete escape, and study the long time survival probability for small holes analytically and numerically. We find that it approaches a universal function of a single parameter, hole area multiplied by time.

I Introduction

Mathematical billiards, in which a particle moves freely while confined in a cavity, are of interest for many physical experiments and applications (where the particle is an atom, electron, photon, etc). Billiards are popular in both mathematics and physics, exhibiting many dynamical features (depending on the cavity shape), with applications from superconductors to the Lorentz gas and ray optics [1]. One useful approach for investigating the dynamics consists of placing one or more holes in the boundary, a random distribution of initial conditions in the billiard, and considering the probability of survival (not escaping) as a function of time. For the single-hole spherical billiard, there is incomplete escape with the survival probability $P_{sc}(t) \not\to 0$ as the time of survival, $t \to \infty$. Here we study the survival probability for long times in an open multi-hole spherical billiard. We provide detailed calculations regarding two billiard configurations, specifically holes centered on the equator of a sphere and a sphere with a thin strip. We show that equal spacing of holes around the equator is an efficient means of ensuring almost complete escape. We also compare this with a thin strip hole circling the equator. Terms that vanish in the long time limit of the survival probability in our configurations comprising almost-complete escape are investigated analytically and numerically. We find that this survival probability approaches a universal function of a single parameter, the total surface area of the hole(s) multiplied by time.

A mathematical billiard is a dynamical system in which a particle is in motion via alternating straight line movements in its interior and mirror-like reflections with its boundary without losing speed [2].

Mathematical billiards are of interest for a diverse collection of examples of dynamical systems (depending on the cavity shape) [3, 4, 5, 6]. Examples of dynamics include chaotic (e.g. brownian motion, Lorentz flows and the Sinai billiard [7]), intermittent, e.g. the drive-belt stadium billiard [8] and regular [2, 9, 10, 11].

We will now consider open dynamical systems, formed by the introduction of a small hole in the boundary or in the interior allowing us to probe their internal dynamical nature. In these systems, the dynamics is no longer considered when reaching the associated hole(s). The survival probability for time t is defined as the chance of a particular trajectory surviving for time t given a suitable distribution of initial conditions. In particular, we are interested in open billiards including a 2-hole stadium [12], billiards with holes in their interior [13] and billiard problems in convex domains [14].

The circular billiard is known for its complete integrability and its relation to a number of well known geometries with chaotic or mixed phase space. We provide an illustration of circle-type billiards in figure 1. For example it is related to the study of mushroom billiards with circular arcs, since circular-billiard orbits occupy the circular part of the mushroom-shaped counterpart's phase space, allowing mushroom billiards to be a visible example of sharply divided phase space [15]. These billiards are widely studied both classically and quantum mechanically [16]. Furthermore, for mushroom billiards, typical values of a control parameter allow the existence of marginally unstable periodic orbits (MUPOs) that exhibit stickiness, specifically that unstable orbits approach regular regions in phase space [17, 18]. In addition, the prevalence of MUPOs in chaotic billiards is related to the use of microwave experiments in annular billiards [19], within which orbits resemble those from the circular billiard. MUPOs are known for their application in the context of directional emission in dielectric microcavities [18]. We will denote the probability of survival for time t, given a uniform distribution of initial conditions, in the circular billiard by $P_c(t)$ (NB: $P_c(t) \not\to 0$ if the hole is in the interior). The density of orbits at the boundary implies that $P_c(t) \to 0$ as $t \to \infty$ since unperturbed periodic orbits constitute a zero-measure set in phase space. Interestingly, the leading coefficient of $P_c(t)$ is related to the Riemann hypothesis (RH) [11], an interesting problem related to number theory [20].

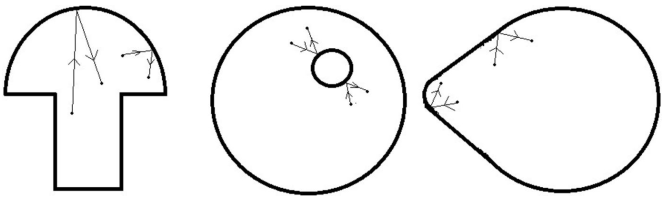

Figure 1: Circle-type billiard dynamics. Mushroom, annulus and drivebelt in the left, middle and right respectively

We stress that open three dimensional billiards have hardly been studied. In our case, we note that no orbits are dense in the sphere since orbits are all contained in planes passing through the centre of the sphere. There are, however, a number of qualitative differences between two and three dimensional billiards. For example, the defocusing phenomenon in spaces of dimensions of at least three is different to that in two dimensions due to the optical phenomenon of astigmatism [21]. The spherical billiard is of particular interest for applications, e.g. whispering gallery mode emission from a spherical microcavity [22], while simple due to its symmetric geometry and hence we expect to easily obtain results for such a configuration. The variation of the rate of decay of such a billiard's survival probability with time is generally dependent on the hole shape, which is trivial in two dimensions, but allows a rich variety of possibilities in three dimensions.

We will now describe the recently completed single-hole spherical billiard work [9]. An interesting discovery in the work presented in [9], is that the $O(\frac{1}{t})$ term in the long-time survival probability expansion associated with the square-shaped-hole configuration generally increases with hole size whereas this is found to decrease in the circular-shaped-hole configuration. In our previous work on the single-hole spherical billiard configurations [9], it was observed that the particle almost always escapes from the circle but not the sphere; here we are interested in situations involving the sphere with almost complete escape. In addition, the expansion of the long-time survival probability for our single-hole spherical billiard (as per [9]) is attributed to the RH via consideration of the relevant integration of circular planes. The relevant analytic results are and/or can be obtained using analogous asymptotic as well as complex-analysis approaches in [11] and [9].

In this current work we study hole configurations with almost complete escape, namely $P(t) \to 0$ as $t \to \infty$ (where $P(t)$ denotes our survival probability associated with the particular configurations in question). We consider two different forms of holes are involved specifically a collection of circular-shaped holes and a thin-strip hole hence an indication of a variation in the rate of decay of the associated survival probabilities. To aid this, we consider a spherical billiard with a thin strip across its equator as an appropriate comparison of the former configuration, specifically both associated survival probabilities' decay rates are inversely proportional to the relevant hole's/holes' total surface area in the limit of the parameter k. The relevant numerical results are obtained using an approach based on simulating trajectories within these spherical billiard configuration extensions and analysing the range of survival times for a large number of trajectories. Furthermore, analogous to a result from [11] we find numerically that the survival probability associated to our multi-hole spherical billiard configuration converges to a universal function of the product of the total surface area of the holes and time. Interestingly, we find that asymptotic results with associated limits for our multi-hole billiard configuration are similar to that in [11], specifically the long-term survival probabilities tending to 0 as time tends to infinity at a rate which is inversely proportional to the total surface area of the holes. This work is extended from and related to the spherical billiard problem with a single hole, for which similar results regarding the long-time survival probability within our multi-hole spherical billiard configuration are found. There are key questions regarding this problem, related to the convergence of the relevant distribution of particles that have not escaped by a particular time to a limiting long-time distribution [13].

The relevant theoretical calculations are presented in Section II. The corresponding numerical results, are presented in Section III. Analogous results corresponding to our thin-strip comparisons are presented in Section IV. Concluding remarks regarding the results of such configurations as well as further related areas of study are provided in Section V.

II Theoretical analysis

II.1 Objective

We describe an approach in establishing a spherical billiard configuration with a multiple number of holes of a particular size, such that the objective of preventing any trajectory of a particle undergoing internal reflections with the boundary from surviving infinitely, apart from a zero-measure set of trajectories, is achieved.

Furthermore, efficiency in terms of minimizing the number of holes to obtain the above objective will be discussed.

II.2 Setup

We start by finding a condition under which a plane (which will contain the trajectory) intersects a circular hole. This is based on parameterizing a plane in our unit sphere $S^2 = \{\underline{r} : |\underline{r}| = 1\}$ by a unit vector normal to it, which we will denote by $\underline{n}$. We will let each hole be identical with radius ϵ and center $\underline{r_0} \in S^2$ which we define as $\{\underline{r} \in S^2 : \cos^{-1}(\underline{r} \cdot \underline{r_0}) \leq \epsilon\}$ (we have no absolute $|\cdot|$ function within the $\cos^{-1}(\cdot)$ function otherwise we have a hole on the opposite side as well), where $(\cdot)$ denotes the dot product of two vectors.

We define spherical polar coordinates, $(x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$. A suitable approach is considering an array of circular planes in the sphere, specifically the collection of planes defined by $\{\phi \in \{\frac{\pi}{N}, \frac{2\pi}{N}, \ldots, \frac{(N-1)\pi}{N}, \pi\}; N \in \mathbb{N}; \theta \in [0, \frac{\pi}{2}]\ \}$, tending towards an infinite set. This set will be sufficiently large when $\frac{\pi}{N} < 2\epsilon \implies N = \lceil \frac{\pi}{2\epsilon} \rceil$. Without loss of generality we can place these holes centered on the equator of the sphere without them overlapping.

We will also use (ϕ, θ) to define a plane by obtaining an expression for θ in terms of ϕ. We let $\phi^{\perp}$ and $\theta^{\perp}$ denote angular displacements of $\underline{n}$.

Furthermore, we introduce a hole in the spherical billiard centered at the point $(\phi^{\epsilon}, \theta^{\epsilon})$ and with radius ϵ. We can see that the angle between the vector normal to the plane, $\underline{n}$, and the vector pointing to the center of the hole, $\underline{h}$, satisfies $\psi \in [\frac{\pi}{2}-\epsilon, \frac{\pi}{2}+\epsilon]$. A key point is that we assume $\epsilon < \frac{\pi}{2}$ to avoid inevitable intersections with the would-be hemispherical hole. We also derive the following expression for $\underline{n} \cdot \underline{h}$:

$$\begin{aligned} \cos\psi &= \underline{n} \cdot \underline{h} \\ &= \cos(\phi^{\perp} - \phi^{\epsilon}) \sin\theta^{\perp} \sin\theta^{\epsilon} + \cos\theta^{\perp} \cos\theta^{\epsilon}, \end{aligned} \tag{1}$$

where $\cos(\phi^{\perp} - \phi^{\varepsilon}) = \cos\phi^{\perp} \cos\phi^{\varepsilon} + \sin\phi^{\perp} \sin\phi^{\varepsilon}$. Therefore, we find the following conditions for the intersection of a circular plane with this hole:

$$\begin{gathered} \tfrac{\pi}{2} - \epsilon < \psi < \tfrac{\pi}{2} + \epsilon \\ \cos(\tfrac{\pi}{2} + \epsilon) < \cos\psi < \cos(\tfrac{\pi}{2} - \epsilon) \\ |\underline{n} \cdot \underline{h}| < \sin\epsilon, \end{gathered} \tag{2}$$

where we use $\cos(\frac{\pi}{2} \pm \epsilon) = \mp \sin\epsilon$.

II.3 Special configurations

II.3.1 Odd number of holes centered at the equator

We will show that an odd number of identical holes in the spherical billiard, centered along its equator, and separated by each of their diameter (a rational multiple of π) allows our objective of all trajectories, other than periodic orbits, escaping. We provide the example of $k = 1$ in figure 2. We define the center of each hole by:

$$(\phi^{\varepsilon}, \theta^{\varepsilon}) = \left(4l\epsilon_k, \frac{\pi}{2}\right), k; l \in \mathbb{N}, l \in \{0, \ldots, 2k\}, \tag{3}$$

where the radius of each of our holes is defined by $\epsilon_k = \frac{\pi}{2(2k+1)}$.

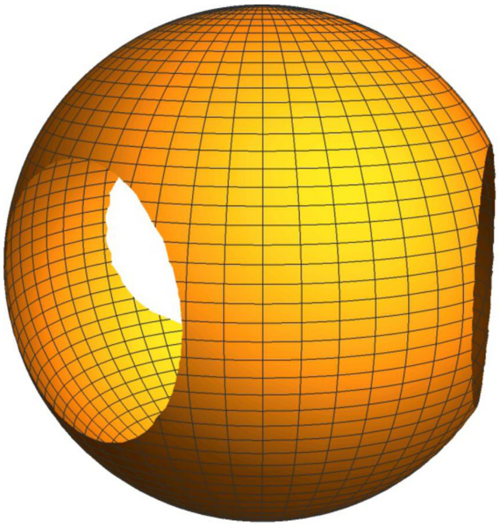

Figure 2: Multi-hole spherical billiard configuration for $k = 1$

Therefore, in this case

$$
\begin{aligned}
\underline{n} \cdot \underline{h} &= \cos\left(\phi^{\perp} - 4l\epsilon_k\right) \sin\theta^{\perp} \sin\left(\frac{\pi}{2}\right) \\
&+ \cos\theta^{\perp} \cos\left(\frac{\pi}{2}\right) \\
&= \sin\left(\frac{\pi}{2} + \phi^{\perp} - 4l\epsilon_k\right) \sin\theta^{\perp},
\end{aligned}
$$

where we can choose $\theta^{\perp} = \frac{\pi}{2}$ to maximize $\underline{n} \cdot \underline{h}$. Hence, $(\frac{\pi}{2} + \phi^{\perp} - 4l\epsilon_k) \in [-\frac{\pi}{2(2k+1)}, \frac{\pi}{2(2k+1)}] \cup [\pi - \frac{\pi}{2(2k+1)}, \pi + \frac{\pi}{2(2k+1)}] \implies \phi^{\perp} \in [\frac{2l\pi}{2k+1} \pm \frac{\pi}{2} - \frac{\pi}{2(2k+1)}, \frac{2l\pi}{2k+1} \pm \frac{\pi}{2} + \frac{\pi}{2(2k+1)}]$ (since $|\sin x| \leq 1$ $\forall x \in \mathbb{R}$). In addition, we can have that $\sin\theta^{\perp} < \frac{\pi}{2(2k+1)} \implies \theta^{\perp} \in [0, \frac{\pi}{2(2k+1)}] \cup [\pi - \frac{\pi}{2(2k+1)}, \pi]$. However, $\cup_{l=0}^{2k}\{[\frac{2l\pi}{2k+1} - \frac{\pi}{2} - \frac{\pi}{2(2k+1)}, \frac{2l\pi}{2k+1} - \frac{\pi}{2} + \frac{\pi}{2(2k+1)}], [\frac{2l\pi}{2k+1} + \frac{\pi}{2} - \frac{\pi}{2(2k+1)}, \frac{2l\pi}{2k+1} + \frac{\pi}{2} + \frac{\pi}{2(2k+1)}]\} = [0, 2\pi)$ and therefore all planes intersect some hole in this configuration.

We illustrate this with an example of 3 equally spaced and identical holes ($\theta^{\epsilon} = \theta^{\frac{\pi}{6}}$). Hence, we have

$$
\left(\phi^{\varepsilon}, \theta^{\varepsilon}\right) = \left(\phi^{\frac{\pi}{6}}, \theta^{\frac{\pi}{6}}\right) = \left\{\left(\frac{2l\pi}{3}, \theta^{\frac{\pi}{6}}\right)\right\},
$$

for $l \in \{0, 1, 2\}$. We try the best and simplest choice of avoiding the holes by selecting $\phi^{\perp} = 0$, since we would have $|\underline{n} \cdot \underline{h}| = |\cos\frac{2l\pi}{3}| = |\sin(\frac{\pi}{2} + \frac{2l\pi}{3})| = |\sin\frac{(3+4l)\pi}{6}| = \sin(\frac{\pi}{6}) \iff l \in \{1, 2\}$ (similarly for other odd numbers of holes). Therefore,

$$
\underline{n} \cdot \underline{h} = \cos\phi^{\frac{\pi}{6}} \sin\theta^{\perp} \sin\theta^{\frac{\pi}{6}} + \cos\theta^{\perp} \cos\theta^{\frac{\pi}{6}}. \tag{4}
$$

We need $l = 1$ (equivalent to $l = 2$ while avoiding intersection with $l = 0$). We have that

$$
\begin{aligned}
\cos\phi^{\frac{\pi}{6}} &= \cos\frac{2\pi}{3} = -\frac{1}{2} \\
\underline{n} \cdot \underline{h} &= \cos\theta^{\perp} \cos\theta^{\frac{\pi}{6}} - \frac{1}{2} \sin\theta^{\perp} \sin\theta^{\frac{\pi}{6}}
\end{aligned}
$$

On the equator $\theta^{\epsilon} = \theta^{\frac{\pi}{6}} = \frac{\pi}{2}$. Therefore,

$$
\begin{aligned}
\underline{n} \cdot \underline{h} &= -\frac{1}{2} \sin\theta^{\perp} \\
|\underline{n} \cdot \underline{h}| &= \sin\theta^{\frac{\pi}{6}} \text{ if } \theta^{\perp} = \frac{\pi}{2}.
\end{aligned}
$$

For all $\theta^{\perp}$, $|\underline{n} \cdot \underline{h}| \leq \sin\frac{\pi}{6}$.

II.3.2 Odd number of holes centered at same height above the equator

We show that the same collection of holes shifted above the equator does not block all planes. In this case, we have $\theta^{\epsilon} < \frac{\pi}{2}$. We involve the use of computing the maximum of $\underline{n} \cdot \underline{h}$ with respect to $\theta^{\perp}$.

$$
\begin{aligned}
\frac{d}{d\theta^{\perp}} \underline{n} \cdot \underline{h} &= \cos(\phi^{\perp} - \phi^{\epsilon}) \cos(\theta^{\perp}) \sin\theta^{\epsilon} \\
&- \sin\theta^{\perp} \cos\theta^{\epsilon} = 0 \\
\implies \theta^{\perp}_{max} &= \tan^{-1}(\cos(\phi^{\perp} - \phi^{\varepsilon}) \tan\theta^{\varepsilon}),
\end{aligned}
$$

which yields the following extremal value of $\underline{n} \cdot \underline{h}$:

$$
\begin{aligned}
\underline{n}\cdot\underline{h}_{max} &= \cos(\theta^{\perp}_{max})\cos\theta^{\varepsilon}\\
&+ \cos(\phi^{\perp}-\phi^{\epsilon})\sin\theta^{\perp}_{max}\sin\theta^{\varepsilon}\\
&= \frac{\cos\theta^{\epsilon}}{\sqrt{\cos^2(\phi^{\perp}-\phi^{\epsilon})\tan^2\theta^{\epsilon}+1}}\\
&+ \cos(\phi^{\perp}-\phi^{\epsilon})\frac{\cos(\phi^{\perp}-\phi^{\epsilon})\tan\theta^{\epsilon}}{\sqrt{\cos^2(\phi^{\perp}-\phi^{\epsilon})\tan^2\theta^{\epsilon}+1}}\sin\theta^{\epsilon}\\
&= \sqrt{\cos^2(\phi^{\perp}-\phi^{\epsilon})+(1-\cos^2(\phi^{\perp}-\phi^{\epsilon}))\cos^2\theta^{\epsilon}}\\
&> |\cos(\phi^{\perp}-\phi^{\epsilon})|
\end{aligned}
$$

since $\theta^{\epsilon}\neq\frac{\pi}{2}$ hence $\cos^2\theta^{\epsilon}\neq 0$ where we use $\cos(\tan^{-1}x)=\frac{1}{\sqrt{x^2+1}}$ and $\sin(\tan^{-1}x)=\frac{x}{\sqrt{x^2+1}}$. Therefore, $\underline{n}\cdot\underline{h}$ is greater than $\sin\Big(\epsilon_k\Big)$ if

$$
\frac{\pi}{2}+\phi^{\perp}-\phi^{\epsilon} \quad\in\quad \Big[\epsilon_k,\pi-\epsilon_k\Big]\cup\Big[\pi+\epsilon_k,2\pi-\epsilon_k\Big],
$$

for some $\phi^{\epsilon}=4l\epsilon_k$ so that a non-zero measured set of planes does not intersect a hole.

We now justify that a plane with a unit normal vector pointing to the center of one of our holes does not intersect any other hole. We have that the relevant dot product in this case is:

$$
\begin{aligned}
\cos\Big(4l_1\epsilon_k\Big)\cos\Big(4l_2\epsilon_k\Big) &+ \sin\Big(4l_1\epsilon_k\Big)\sin\Big(4l_2\epsilon_k\Big)\\
&= \cos\Big(4(l_1-l_2)\epsilon_k\Big)\\
&= \sin\Big(\frac{\pi}{2}+4(l_1-l_2)\epsilon_k\Big),
\end{aligned}
$$

from which we find that if it is assumed that $n\pi-\epsilon_k\leq\left(\frac{\pi}{2}+4(l_1-l_2)\epsilon_k\right)\leq n\pi+\epsilon_k$ for some integer n we can find that $-1\leq(2k+1)(1-2n)+4(l_1-l_2)\leq 1$ which is only possible if $|(2k+1)(1-2n)+4(l_1-l_2)|=1$ since $(2k+1)(1-2n)+4(l_1-l_2)$ is an odd integer.

To illustrate the general derivations above, we now consider the case of $\theta^{\epsilon}=\theta^{\frac{\pi}{6}}<\frac{\pi}{2}$, $(2k+1=3,\epsilon=\frac{\pi}{6})$.

$$
\begin{aligned}
\frac{d}{d\theta^{\perp}}\underline{n}\cdot\underline{h} &= -\sin\theta^{\perp}\cos\theta^{\frac{\pi}{6}}-\frac{1}{2}\cos\theta^{\perp}\sin\left(\frac{\pi}{6}\right)\\
&= 0\\
\implies \theta^{\perp} &= \tan^{-1}\left(-\frac{1}{2}\tan\theta^{\frac{\pi}{6}}\right),
\end{aligned}
$$

which yields the following extremal value of $\underline{n}\cdot\underline{h}$:

$$
\begin{aligned}
\underline{n}\cdot\underline{h} &= \cos\left(\tan^{-1}\left(\frac{1}{2}\tan\theta^{\frac{\pi}{6}}\right)\right)\cos\theta^{\frac{\pi}{6}}\\
&+ \frac{1}{2}\sin\left(\tan^{-1}\left(\frac{1}{2}\tan\theta^{\frac{\pi}{6}}\right)\right)\sin\theta^{\frac{\pi}{6}}\\
&= \frac{\cos\theta^{\frac{\pi}{6}}}{\sqrt{\frac{1}{4}\tan^2\theta^{\frac{\pi}{6}}}}+\frac{1}{2}\frac{\frac{1}{2}\tan\theta^{\frac{\pi}{6}}}{\sqrt{\frac{1}{4}\tan^2\theta^{\frac{\pi}{6}}}}\sin\theta^{\frac{\pi}{6}}\\
&= \sqrt{\frac{1}{4}+\frac{3}{4}\cos^2\theta^{\frac{\pi}{6}}}>\sin\left(\frac{\pi}{6}\right)
\end{aligned}
$$

for any $\theta^{\frac{\pi}{6}}\neq\frac{\pi}{2}$. Therefore, we cannot establish the result of blocking all planes by shifting above the equator. We provide an alternative way of justifying this.

We note that for $\theta^{\perp},\theta^{\epsilon}\in[0,\pi]$, $\cos(\phi^{\perp}-\phi^{\epsilon})\geq -1$. Therefore $\cos\psi\geq\cos(\theta^{\perp}+\theta^{\epsilon})$. Hence, $\cos\psi\geq\sin\epsilon \iff \theta^{\perp}+\theta^{\epsilon}\in[0,\frac{\pi}{2}-\epsilon]\cup[\frac{3\pi}{2}+\epsilon,2\pi] \implies \theta^{\perp}\in[-\theta^{\epsilon},\frac{\pi}{2}-\epsilon-\theta^{\epsilon}]\cup[\frac{3\pi}{2}+\epsilon-\theta^{\epsilon},2\pi-\theta^{\epsilon}]$. However, without loss of generality, we assume that $\theta^{\perp}\in[0,\pi]$. Therefore, we have that $\theta^{\perp}\in[0,\frac{\pi}{2}-\epsilon-\theta^{\epsilon}]$. Therefore, we have found a domain of values for which there are planes that do not intersect any hole centered above the equator of the sphere.

III Numerical results

We begin this section by finding out the effect (if any) of increasing the number, $2k+1$ ($k\in\mathbb{N}$), of equally spaced holes each with corresponding radius $\frac{\pi}{2(2k+1)}$ centered on the equator on the survival probability $P_{s,mhs}(t)$. We present this in figure 3 for $k\in\{10,100,1000\}$ as well as for the case of 10^8 samples and a maximum time limit of 10^5.

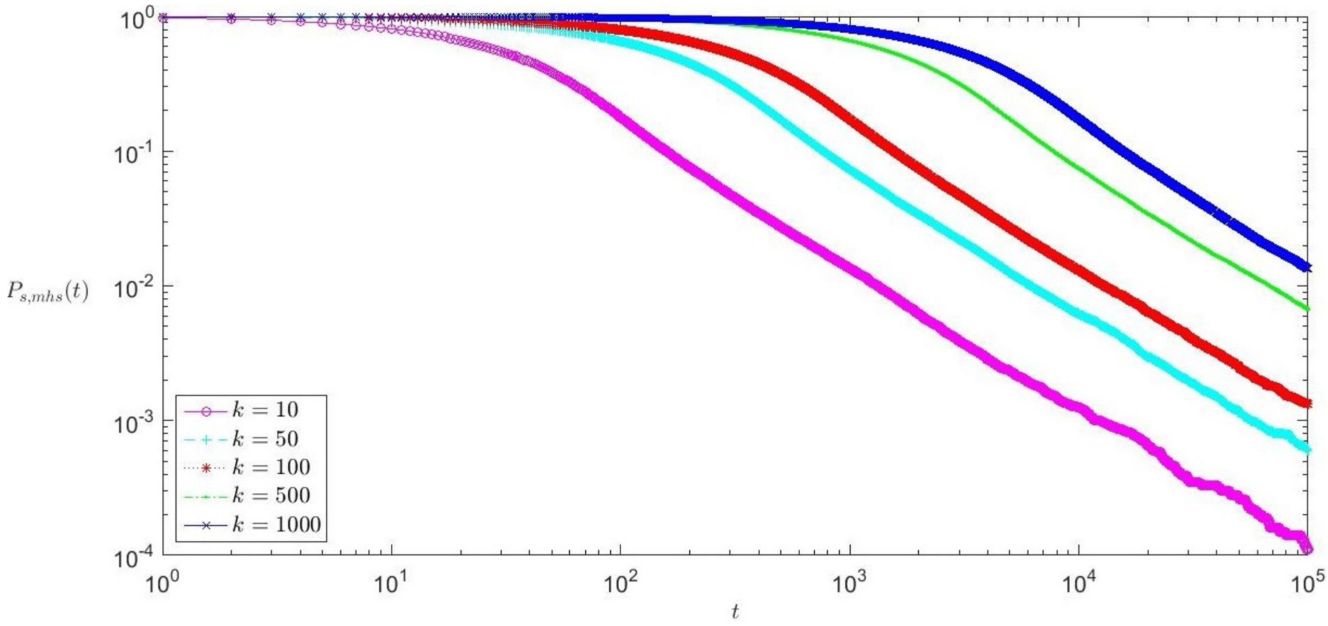

Figure 3: Effect of k on $P_{s,mhs}(t)$, at large t, $P_{s,mhs}(t) \sim \frac{const}{t}$

From the above plot, there is an indication that $P_{s,mhs}(t)$ increases with k, since the total area of the holes decreases with k.

We now describe numerical results for approximation of $P_{s,mhs}(t)$ for small hole sizes as well as large times of survival. We find that initial conditions are less probable in surviving for a large time.

Due to the geometry of holes resulting in a zero-measure set of planes that does not intersect a hole, we select fits that each have 0 as their constant coefficient. We observe that the leading term in $P_{s,mhs}(t)$ as $t \to \infty$ is $\frac{const}{t}$ and so define

$$B_{s,mhs} = \lim_{t\to\infty} tP_{s,mhs}(t) \tag{5}$$

hence use fits of the form

$$P_{s,mhs}(t) \approx \frac{B}{t} + \frac{C}{t^2} \tag{6}$$

Furthermore, for large k, the total area $A_{s,mhs}(k) = 2\pi(2k+1)\left(1 - \cos\left(\frac{\pi}{2(2k+1)}\right)\right) \approx \frac{\pi^3}{8k}$.

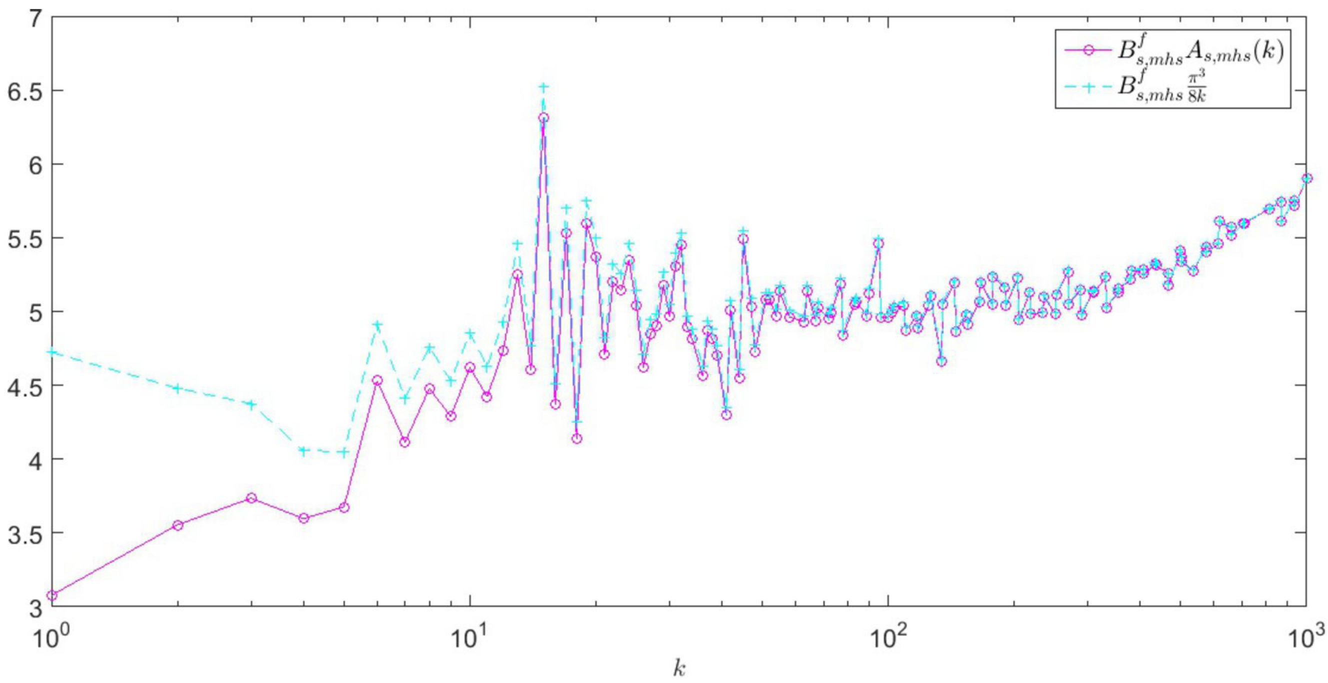

Figure 4: Plot of $B_{s,mhs}A_{s,mhs}(k)$ vs k and $\frac{B_{s,mhs}\pi^3}{8k}$ vs k

We are provided with an indication that $B_{s,mhs}A_{s,mhs}(k)$ tends to a constant as $k \to \infty$.

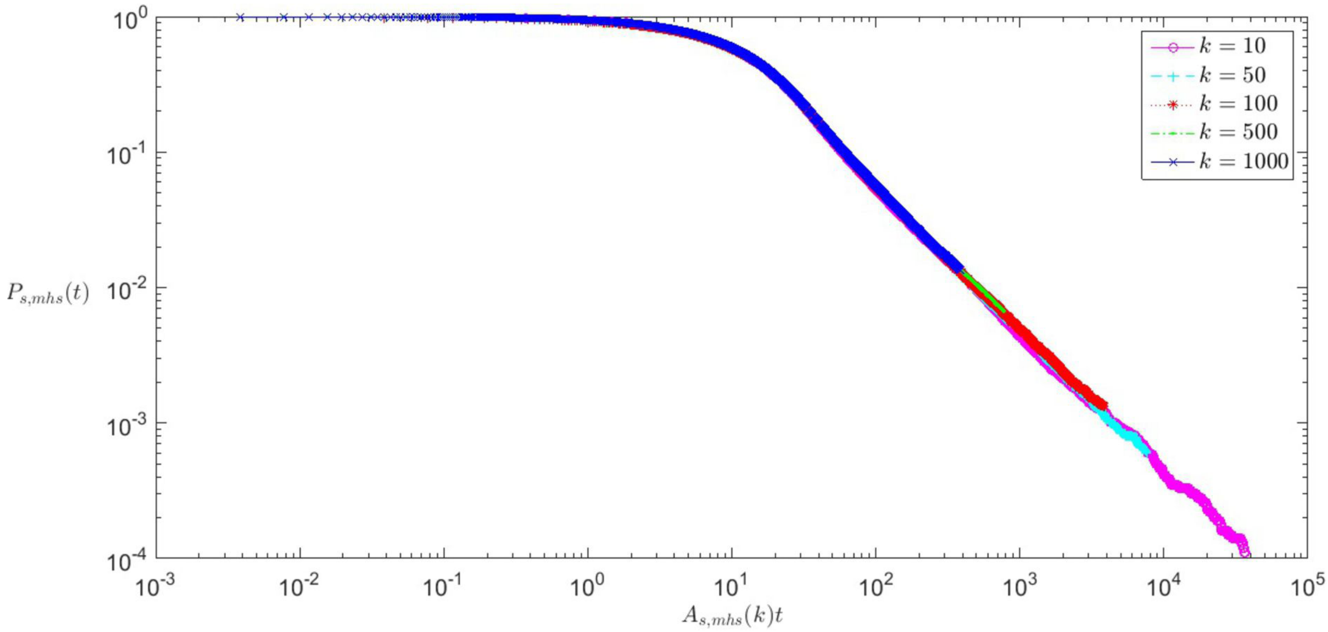

Figure 5: $P_{s,mhs}(t)$ vs $A_{s,mhs}(k)t$

From figure 5, we see that $P_{s,mhs}(t)$ follows an $O\left(\frac{1}{A_{s,mhs}(k)t}\right)$ trend for large $A_{s,mhs}(k)t$. In general there is a scaling to a universal function of $A_{s,mhs}(k)t$ in the limits of large time and small hole size hence an analogous trend to that in [10, 11].

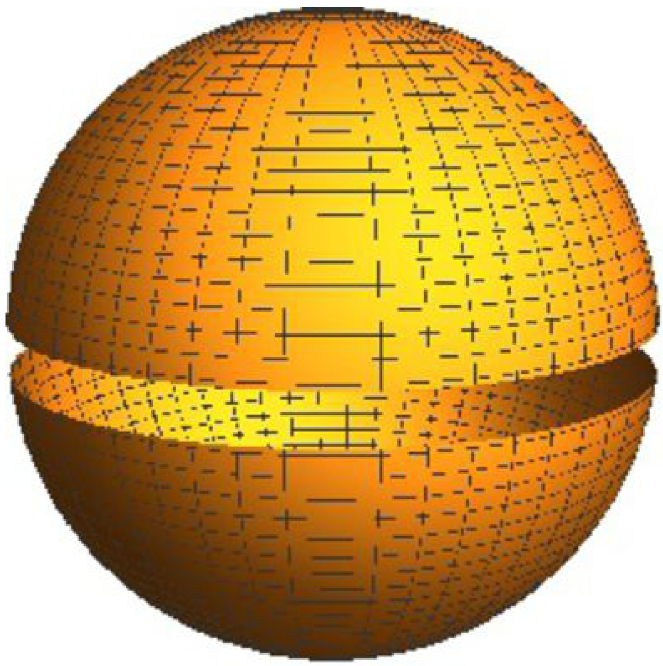

Figure 6: Spherical billiard with a thin strip

IV Thin-strip comparison model

We now describe an analogous configuration in the limit of large k (hence in the limit of a large number of holes, $2k+1$).

We will theoretically justify that asymptotic results for the case of our special multi-hole configuration as well as this thin-strip configuration (the approximate limiting hole shape) approximately agree in the limit in question. For brevity we define the radius of each of our holes as well as the corresponding rectangle by $\epsilon_k = \frac{\pi}{2(2k+1)}$. Specifically, we have a configuration of a thin strip defined by $\{\phi \in [0, 2\pi], \theta \in [\frac{\pi}{2} - \epsilon_k, \frac{\pi}{2} + \epsilon_k]\}$, where ϕ denotes the angular displacement of a point on the sphere with respect to the positive x-axis and θ denotes its angular displacement with respect to the positive z-axis. We explain further by recalling that billiard orbits in the spherical billiard lie on planes. Therefore, our consideration of an orbit is reduced to an open circular billiard problem. The resulting size of each hole on each side of a circular plane inside our thin-strip configuration with particular orientation for particular θ is

$$\begin{aligned} h &= \pi - 2\cos^{-1}\left(\frac{\cos\left(\frac{\pi}{2}-\epsilon_k\right)}{\cos\theta}\right) \\ &= \pi - 2\cos^{-1}\left(\frac{\sin\left(\epsilon_k\right)}{\cos\theta}\right). \end{aligned} \tag{7}$$

Furthermore, the dominant contributions to the survival probability come from periodic orbits of rotation number m and period n, which are enumerated with m and $l = \lceil \frac{n}{2} \rceil$. We also have that m and n are coprime: $(m, n) = 1$. We recall that numerically the survival probability for large time t in our multi-hole configuration is $P_{s,mhs}(t) = O(\frac{k}{t})$ and provide a brief justification of this which is analogous to the single hole sphere in [9]. In the derivation of equation (8) from [9], we split the sum over n (the period of trajectories in an open circular billiard that never escape and which are perturbed by a small amount) into two parts, over odd numbers and even numbers so that for time t, and orbit parameters l and m and each identical hole's size h, the asymptotic survival probability for small h in a circular billiard contained within a spherical billiard with a thin-strip hole is

$$\begin{aligned} P_{c,str}(t) &= \frac{3}{2\pi t}\left[\sum_{l=1}^{\infty}\left(\sum_{\substack{m=0 \\ (m,2l)=1}}^{l-1} 2lG\Big(\frac{\pi}{l}-h\Big)\sin^3\frac{\pi m}{2l}\cos\frac{\pi m}{2l}\right.\right. \\ &+ \left.\left.\sum_{\substack{m=0 \\ (m,2l-1)=1}}^{l-1} (2l-1)2G\Big(\frac{\pi}{2l-1}-h\Big)\sin^3\frac{\pi m}{2l-1}\cos\frac{\pi m}{2l-1}\right)\right], \\ &\approx \frac{3}{2\pi t}\left[\int_0^{\frac{\pi}{h}}\frac{2}{\pi^2}\int_0^{l} 2l\Big(\frac{\pi}{l}-h\Big)^2\sin^3\frac{\pi m}{2l}\cos\frac{\pi m}{2l}dmdl\right. \\ &+ \left.\int_0^{\frac{\pi}{2h}+\frac{1}{2}}\frac{4}{\pi^2}\int_0^{\frac{2l-1}{2}}(2l-1)2\Big(\frac{\pi}{2l-1}-h\Big)^2\sin^3\frac{\pi m}{2l-1}\cos\frac{\pi m}{2l-1}dmdl\right] \\ &\approx \frac{3}{\pi h t} \end{aligned} \tag{8}$$

as $h \to 0$, where we use $\frac{2}{\pi^2}$ as the asymptotic probability that two positive integers are coprime as well as that their maximum is even, $\frac{4}{\pi^2}$ as the asymptotic probability that two positive integers are coprime as well as that their maximum is odd (see Appendices A and B) and let θ' denote the angular displacement between two identical holes if they were exactly in the same position $\pmod{\frac{2\pi}{n}}$ as well as

$$\theta' = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{\pi}{n} & \text{if } n \text{ is odd.} \end{cases} \pmod{\frac{2\pi}{n}} \tag{9}$$

(which appears in $G(\theta' - h)$ and $G(\frac{2\pi}{n} - \theta' - h)$ from equation (2) in [11]) and

$$G(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{10}$$

We stress that the expression in equation (8) is significantly different from reference [9] because each circular billiard now has two symmetrically placed holes rather than one.

From integrating equation (8) over circular planes, introducing the transformation $\theta_P = (\frac{\pi}{2} - \epsilon_k)\tau_k$ (where θ_P denotes the angular displacement of a circular plane from the positive z-axis), expanding a circular plane's corresponding hole size $h = \pi - 2\cos^{-1}\left(\frac{\cos\left(\frac{\pi}{2}-\epsilon_k\right)}{\cos\theta_P}\right) = \frac{2\epsilon_k}{\cos\left(\frac{\pi}{2}\tau_k\right)} + \ldots$ for small ϵ_k, integrating equation (8) from $\tau_k = 0$ to $\tau_k = 1$ (similar to [9]) as well as taking leading order terms, the corresponding asymptotic survival probability for the spherical billiard with a thin strip as $\epsilon_k \to 0$ is

$$P_{s,str}(t) \sim \frac{3k}{2\pi t} \tag{11}$$

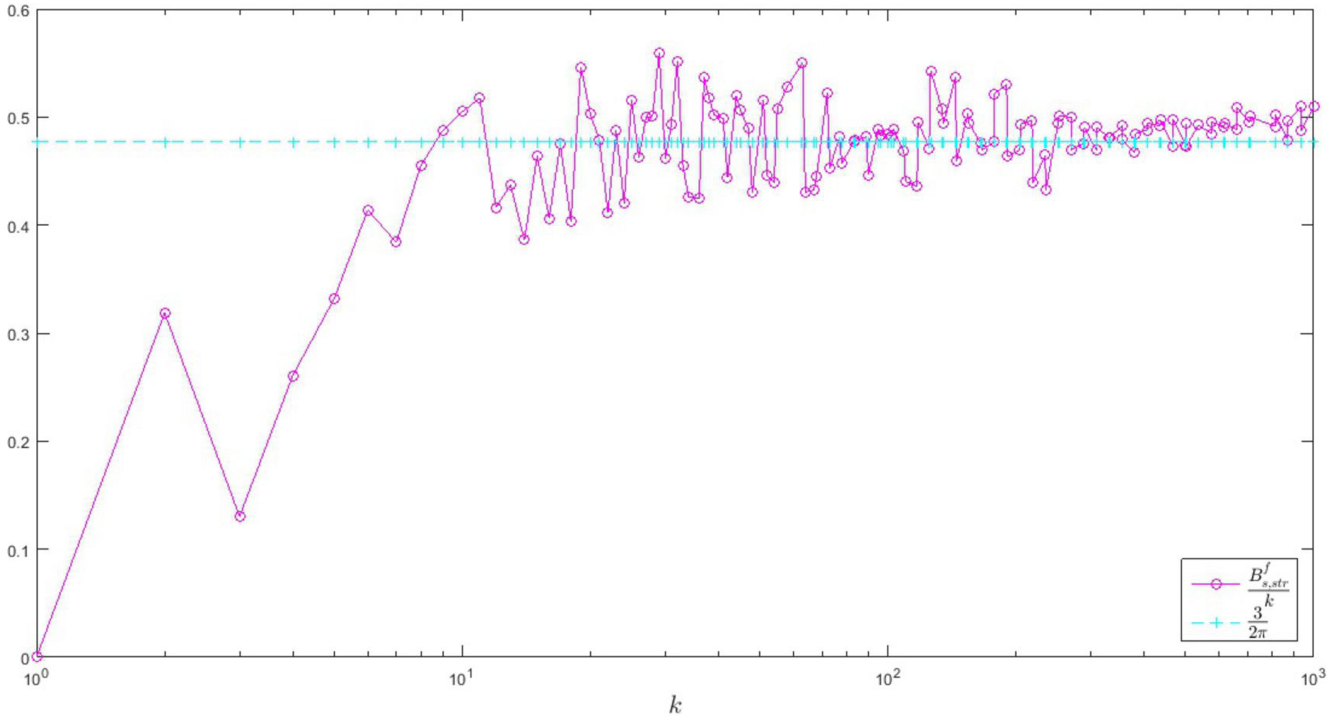

Figure 7: Spherical billiard with a thin strip's confirmation of equation (11)

We also provide additional numerical results in support of equation (11) in figure 8. In this case, we are provided with indication that our numerical simulation curves for various values of k converge to equation (11) in the limit of $k \to \infty$.

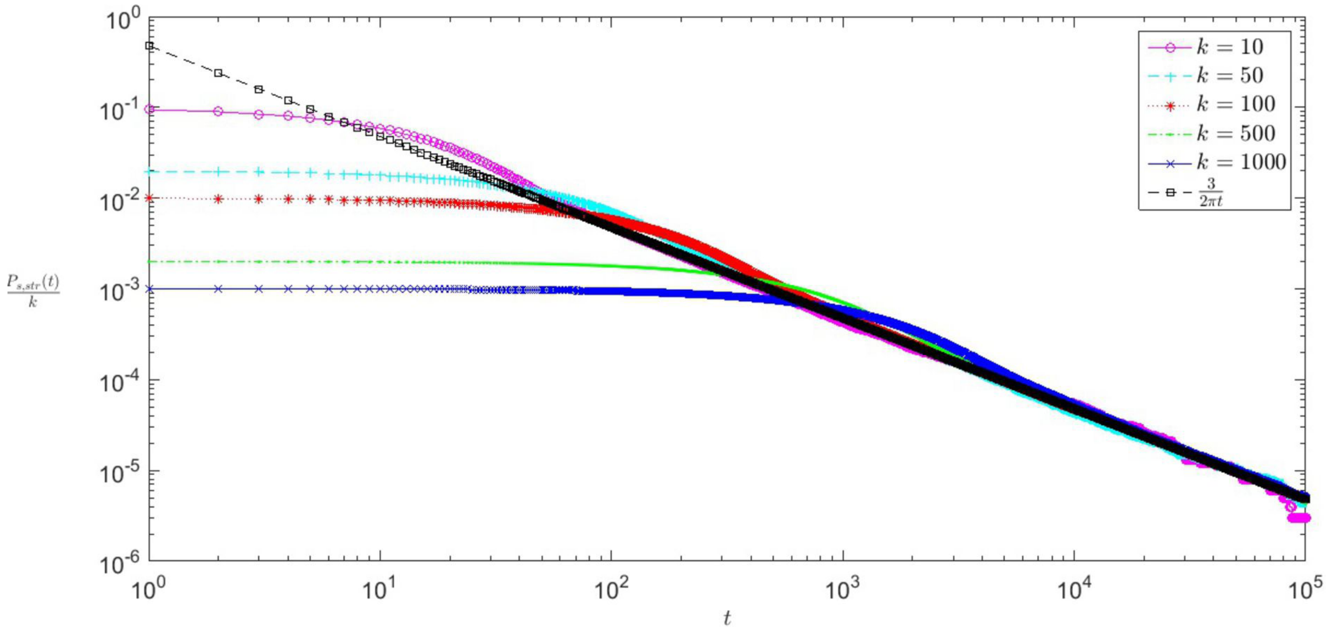

Figure 8: Effect of k on $P_{s,str}(t)$ at large t compared with our analytical result in equation (11)

We define $A_{s,str}(k) = 4\pi \sin\left(\frac{\pi}{2(2k+1)}\right)$ as the surface area of our thin-strip configuration's strip-shaped hole. Furthermore, we find that $A_{s,str}(k) \approx \frac{\pi^2}{k}$ as $k \to \infty$. Hence we have the following approximation:

$$P_{s,str} \sim \frac{3k}{2\pi t} \approx \frac{3\pi}{2A_{s,str}(k)t}. \tag{12}$$

V Conclusion

In this work, we have described and attained an objective of establishing a multi-hole billiard configuration that prevents only a zero-measure set of trajectories from escaping. In particular, we can see that the long-time survival probability for our multi-hole billiard configuration is approximately inversely proportional to the product of the total surface area of the holes and time. Furthermore, we can see from the analysis and calculations undertaken that the number of holes of a particular size needed to fulfill this objective decreases with increasing hole size as well as that the survival probability for our multi-hole billiard configuration increases as k increases.

We have produced calculations regarding special spherical-billiard configurations, specifically:

1) An odd number of holes centered on the equator (from which we find that the objective is fulfilled);

2) An odd number of holes centered at the same height above the equator (from which we find that not all arbitrarily selected circular planes within the spherical billiard intersect a hole) and;

3) A Thin-Strip configuration.

Further potential areas of study include multi-hole billiard configurations with non-circular holes (e.g. the square-shaped hole in [9]); multi-hole billiard configurations with holes placed at asymmetric locations; multi-hole billiard configurations with holes of varying radii; time-dependent thin-strip holes; and escape and transmission survival probabilities (injection through one hole and escape through another) for a multi-hole billiard configuration (similar to [12]). The study of time-dependent holes seems most interesting since the survival probability can vary irregularly [23].

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A The probability that m and n are coprime given that $max(m,n)$ is even

We first need the sum $\sum_{d \text{ odd}} \frac{\mu(d)}{d^2}$ which we find as follows

$$\begin{aligned}
\sum_{d \text{ even}} \frac{\mu(d)}{d^2} &= -\frac{1}{2^2} \sum_{d \text{ odd}} \frac{\mu(d)}{d^2} \\
\implies \sum_{d} \frac{\mu(d)}{d^2} &= \frac{3}{4} \sum_{d \text{ odd}}, \\
\sum_{d} \frac{\mu(d)}{d^2} &= \frac{6}{\pi^2} \\
\implies \sum_{d \text{ odd}} = \tfrac{4}{3} \cdot \tfrac{6}{\pi^2} &= \frac{8}{\pi^2}
\end{aligned}$$

Without loss of generality as well as for simplicity we consider pairs (m,n) such that $m \leq n$. We first provide the following calculation of the asymptotic probability that two integers are coprime given that their maximum is even:

$$\begin{aligned}
\frac{\sum_{i=1}^{l} \sum_{d|(2i)}^{2i} \mu(d) \sum_{\substack{m=d \\ d|m}}^{2i} 1}{2l + 2(l-1) + \ldots + 2} &= \frac{\sum_{i=1}^{l} \sum_{d|(2i)}^{2i} \mu(d) \frac{2i}{d}}{l(l+1)} = \frac{\sum_{i=1}^{l} \sum_{\substack{d|(2i) \\ d \text{ odd}}}^{2i} \mu(d) \frac{2i}{d}}{l(l+1)} \\
&+ \frac{\sum_{i=1}^{l} \sum_{\substack{d|(2i) \\ d \text{ even}}}^{2i} \mu(d) \frac{2i}{d}}{l(l+1)} = \frac{\sum_{d \text{ odd}}^{l} \mu(d) 2 \frac{d(1+\ldots+\lfloor \frac{l}{d} \rfloor)}{d}}{l(l+1)} \\
&+ \frac{\sum_{d'=1}^{l} \mu(2d') \frac{d'(1+\ldots+\lfloor \frac{l}{d'} \rfloor)}{d'}}{l(l+1)} \underset{l \to \infty}{\to} \sum_{d \text{ odd}} \frac{\mu(d)}{d^2} - \frac{1}{2} \sum_{d' \text{ odd}} \frac{\mu(d')}{d'^2} \\
&= \frac{4}{\pi^2},
\end{aligned}$$

where $\mu(\cdot)$ denotes the Möbius function and $d = 2d'$ in the case of d even. We now provide the following calculation for the asymptotic probability that two integers are coprime as well as that their maximum is even:

$$\frac{4}{\pi^2}\cdot\frac{2l+2(l-1)+\ldots+2}{2l+2l-1+\ldots+1} = \frac{4}{\pi^2}\cdot\frac{l(l+1)}{\frac{1}{2}\cdot 2l\cdot(2l+1)} \underset{l\to\infty}{\to} \frac{2}{\pi^2}$$

B The probability that m and n are coprime given that $max(m,n)$ is odd

Similar to the previous appendix, we provide the following calculation of the asymptotic probability that two integers are coprime given that their maximum is odd:

$$\begin{aligned}
\frac{\sum_{i=1}^{l}\sum_{d|(2i-1)}^{2i-1}\mu(d)\sum_{\substack{m=d\\ d|m}}^{2i-1}1}{2l-1+2(l-1)-1+\ldots+1} &= \frac{\sum_{i=1}^{l}\sum_{d|(2i-1)}^{2i}\mu(d)\frac{2i-1}{d}}{l^2}\\
&= \frac{\sum_{i=1}^{l}\sum_{\substack{d|(2i-1)\\ d\text{ odd}}}^{2i-1}\mu(d)\frac{2i-1}{d}}{l^2}\\
&= \frac{\sum_{d\text{ odd}}^{2l-1}\mu(d)\frac{d(1+3+\ldots+\lfloor\frac{2l-1}{d}\rfloor)}{d}}{l^2}\\
&\underset{l\to\infty}{\to} \sum_{d\text{ odd}}\frac{\mu(d)}{d^2}\\
&= \frac{8}{\pi^2},
\end{aligned}$$

and similar to the even case, we provide the following calculation for the asymptotic probability that two integers are coprime as well as that their maximum is odd:

$$\frac{8}{\pi^2}\cdot\frac{2l-1+2(l-1)-1+\ldots+1}{2l+2l-1+\ldots+1} = \frac{8}{\pi^2}\cdot\frac{l^2}{\frac{1}{2}\cdot 2l\cdot(2l+1)} \underset{l\to\infty}{\to} \frac{4}{\pi^2}$$

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