Mathematical Methods 3	**2006**

Sheet D — Fourier Transforms

In this worksheet we consider various properties of Fourier transforms, including convolutions.

1. The Fourier transform of $f(x)$ is assumed to exist and is given to be $F(k)$. Verify the Fourier transforms $G(k)$ of the following functions $g(x)$ in terms of F.

   (a) $g(x) = f(ax) \Rightarrow G(k) = \dfrac{1}{|a|}F(k/a)$

   (b) $g(x) = f(x-a) \Rightarrow G(k) = \mathrm{e}^{+\mathrm{i}ak}F(k)$

   (c) $g(x) = \mathrm{e}^{-ax}f(x) \Rightarrow G(k) = F(k+\mathrm{i}a)$

   (d) $g(x) = \mathrm{e}^{\mathrm{i}ax}f(x) \Rightarrow G(k) = F(k+a)$

2. (a) Show that the Fourier Transform of $xy(x)$ is $(-i)\dfrac{dY(k)}{dk}$ where $Y(k)$ is given to be the Fourier Transform of $y(x)$.

   (b) Find the Fourier transform, $Y(k)$, of the 'Airy equation', $y''(x) - xy(x) = 0$, where you are given that $Y(0) = 1$. Hence show that

$$y(x) = \frac{1}{\pi}\int_0^\infty \cos(\tfrac{1}{3}k^3 + kx)\,\mathrm{d}k$$

   This solution is called an Airy function, $\mathrm{Ai}(x)$.

3. Using Parseval's relation:

$$\int_{-\infty}^{\infty} |h(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(k)|^2\, dk$$

   for $h(x) = f(x) \pm g(x)$ and $h(x) = f(x) \pm \mathrm{i}g(x)$, derive the **general form**:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\overline{G(k)}dk$$

   where the overbar denotes the complex conjugate.

4. (a) From the lectures, we know that for a function $f(x) = \mathrm{e}^{-bx}$ where $b > 0$, the Fourier sine and cosine transforms are given by $F_s(k) = k/(k^2+b^2)$ and $F_c(k) = b/(k^2+b^2)$. Use these results to show that

$$\mathcal{F}_c\{\mathrm{e}^{-x}\cos x\} = \frac{k^2+2}{k^4+4}, \qquad \mathcal{F}_s\{\mathrm{e}^{-x}\cos x\} = \frac{k^3}{k^4+4},$$

   (b) Hence evaluate the integral $\displaystyle\int_0^\infty \frac{x^2+2}{x^4+4}dx$

   (c) Also use the information in part (a) to find $\mathcal{F}_c\{x\mathrm{e}^{-x}\}$ and $\mathcal{F}_s\{x\mathrm{e}^{-x}\}$.

5. From the definitions of the Fourier Sine and Cosine transforms verify that

$$\int_0^\infty f(x)g(x)dx = \frac{2}{\pi}\int_0^\infty F_c(k)G_c(k)dk = \frac{2}{\pi}\int_0^\infty F_s(k)G_s(k)dk$$

6. Prove the convolution result for Fourier Cosine Transforms. That is, if $H_c(k) = G_c(k)F_c(k)$ where $F_c(k) = \mathcal{F}_c\{f(x)\}$, $G_c(k) = \mathcal{F}_c\{g(x)\}$ and $H_c(k) = \mathcal{F}_c\{h(x)\}$ then

$$h(x) = \tfrac{1}{2}\int_0^\infty g(\xi)[f(x+\xi) + f(x-\xi)]d\xi$$

where $f(x)$ is extended as an even function into $x < 0$. [HINT: Follow lecture notes for the convolution proof for F.T's].

7. Show, using double integrals and polar co-ordinates that

$$I = \int_0^\infty \mathrm{e}^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}, \qquad a > 0$$