Internal and external resonances of dielectric disks

2009 EPL 87 34003

(http://iopscience.iop.org/0295-5075/87/3/34003)

A Letters Journal Exploring the Frontiers of Physics

August 2009

EPL, **87** (2009) 34003
doi: 10.1209/0295-5075/87/34003

www.epljournal.org

Internal and external resonances of dielectric disks

C. P. Dettmann[1], G. V. Morozov[1,3(a)], M. Sieber[1] and H. Waalkens[1,2]

[1] *Department of Mathematics, University of Bristol - Bristol BS8 1TW, UK, EU*
[2] *Department of Mathematics, University of Groningen - Nijenborgh 9, 9747 AG Groningen, The Netherlands, EU*
[3] *Thin Film Centre, University of the West of Scotland - Paisley PA1 2BE, UK, EU*

received 27 March 2009; accepted in final form 23 July 2009
published online 27 August 2009

PACS 42.55.Sa – Microcavity and microdisk lasers
PACS 42.25.-p – Wave optics
PACS 03.50.De – Classical electromagnetism, Maxwell equations

Abstract – Circular microresonators (microdisks) are micron size dielectric disks embedded in a material of lower refractive index. They possess modes with complex eigenvalues (resonances) which are solutions of analytically given transcendental equations. The behavior of such eigenvalues in the small opening limit, *i.e.* when the refractive index of the cavity goes to infinity, is analyzed. This analysis allows one to clearly distinguish between internal (Feshbach) and external (shape) resonant modes for both TM and TE polarizations. This is especially important for TE polarization for which internal and external resonances can be found in the same region of the complex wave number plane. It is also shown that for both polarizations, the internal as well as external resonances can be classified by well-defined azimuthal and radial modal indices.

Introduction. – Thin dielectric microcavities of various shapes filled with a homogeneous material are key components for the construction of optical microresonators and microlasers [1,2]. Their eigenmodes (resonances) are characterized by complex wave numbers $k = k_r + ik_i$, which are complicated solutions of 3D Maxwell equations. However, the modes of microcavities, with the thickness only a small fraction of the mode wavelength, can be studied in 2D formulation with the aid of an effective refractive index n_{eff} which takes into account the material as well as the thickness of the cavity, see, for example, appendix I of ref. [3], or chapter II of ref. [4]. Among such microcavities, circular cavities are one of the few cases where the transcendental equations for complex eigenmodes (resonances) can be found analytically.

The resonances can be divided into two classes. Following the common terminology, see, for example, refs. [5,6], we will call them "internal" (or "Feshbach") and "external" (or "shape") resonances. These notions have their origin in quantum reactions where internal resonances correspond to resonances which become real (bound states) and external resonances stay complex in some limit of the coupling between the reaction coordinate and the internal degrees of freedom. In the context of dielectric microcavities the analogous limit is the so-called small opening limit where the refractive index n of the microcavity goes to infinity. The internal resonances are resonances which become real as $n \to \infty$, while the external resonances are resonances which stay complex (not real) in this limit.

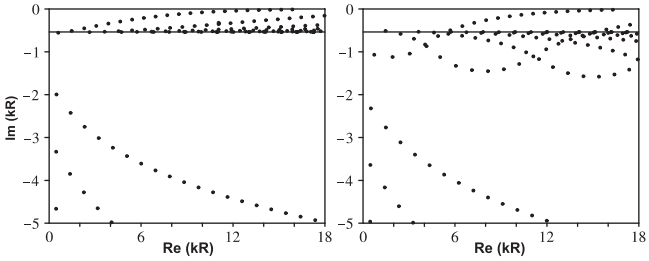

Fig. 1: TM (left panel) and TE (right panel) resonances of a dielectric microdisk of radius R and refractive index $n = 1.5$ in the complex kR plane. The straight solid lines in both panels show the boundary $\operatorname{Im} kR = -\frac{1}{2n} \ln \frac{n+1}{n-1}$.

For transverse magnetic polarization of the electromagnetic field (TM; electric field perpendicular to the disk plane) and a fixed refractive index n, the two kinds of resonances are well separated in the complex wave number plane. As illustrated in the left panel of fig. 1, TM internal resonances with relatively large $\operatorname{Re} kR$ satisfy the inequality, see, for example, refs. [5,6],

$$|\operatorname{Im} kR| \leqslant \frac{1}{2n} \ln \frac{n+1}{n-1}, \tag{1}$$

where R is the radius of the circular disk. Results on scattering at obstacles with convex boundaries, see, for

(a) E-mail: gregory.morozov@uws.ac.uk

example, refs. [7,8], suggest that TM external resonances satisfy

$$|\mathrm{Im}\, kR| > \mathrm{const}\ |\mathrm{Re}\, kR|^{1/3}. \tag{2}$$

Therefore, TM external resonances have in general quite a large imaginary part compared to internal ones. As a consequence they are very leaky (*i.e.* have low Q-factors defined as $Q = k_r/2|k_i|$) and cannot be directly used for lasing.

As illustrated in the right panel of fig. 1, for transverse electric polarization (TE; magnetic field perpendicular to the disk plane) one does not always find an analogous separation. While one can argue that TE external resonances still satisfy a condition similar to (2), there is no condition analogous to (1) for TE internal resonances. For a fixed refractive index n, there is therefore no clear separation between some of the TE internal and external resonances, and indeed, as we will show in this paper, the TE resonances with $\mathrm{Im}\, kR$ in the range from -0.7 to -1.5 in the right panel of fig. 1 consist of a mixture of internal and external ones.

The purpose of this letter is to provide a detailed study of the internal and external circular cavity (disk) resonances. To this end, we carefully study the behavior of the resonances in the small opening limit $n \to \infty$. We note that for internal resonances this has recently been studied in ref. [9]. For completeness we reproduce their results, using however a mathematically different and more illustrative approach. Our analysis allows us to clearly distinguish between internal and external resonant modes for both TM and TE polarizations of the electromagnetic field. This is especially important for TE polarization for which, as we have already mentioned, internal and external resonances can be found in the same region of the complex wave number plane. Moreover, using the small opening limit, we show that both internal and external resonances can be classified by well-defined azimuthal and radial modal indices for both polarizations.

Equations for resonances. – Let Ψ stand for E_z in the case of TM polarization and for H_z in the case of TE polarization, where E_z and H_z are electric and magnetic fields, respectively. For a homogeneous dielectric microdisk of radius R and effective refractive index n in a medium of refractive index 1, Maxwell's equations reduce to

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\varphi^2} + k^2 n^2 \Psi\,(r,\varphi) = 0, \tag{3}$$

inside the microdisk ($r < R$) and the same form with n replaced by 1 outside the microdisk ($r > R$). The resonances are obtained by imposing outgoing boundary conditions at infinity, *i.e.* we require that $\Psi(r) \propto e^{ikr}/\sqrt{r}$, $r \to \infty$. For physical reasons, the value of the EM field at the disk center must be finite. These boundary conditions in combination with the continuity of the electric field E_z and its derivative for TM modes (or the magnetic field H_z and its derivative divided by the square of the refractive index for TE modes) at $r = R$ lead to the resonant field Ψ in the form of twofold degenerate (for $m > 0$) whispering gallery (WG) modes

$$\Psi_z^m = \begin{cases} N_m J_m\,(knr) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix}, & r < R, \\ H_m\,(kr) \begin{pmatrix} \cos m\varphi \\ \sin m\varphi \end{pmatrix}, & r > R, \end{cases} \tag{4}$$

where for TM modes the complex wave numbers k are solutions of

$$J_m(knR)H'_m(kR) - n\, J'_m(knR)H_m(kR) = 0, \tag{5}$$

and for TE modes the complex wave numbers k are solutions of

$$J_m(knR)H'_m(kR) - \frac{1}{n}J'_m(knR)H_m(kR) = 0. \tag{6}$$

Here J_m and H_m are Bessel and Hankel functions of the first kind, respectively, $m = 0, 1, 2, \ldots$ is the azimuthal modal index, and $N_m = H_m(kR)/J_m(knR)$ are constants. Physically, the azimuthal modal index m characterizes the field variation along the disk circumference, with the number of intensity hotspots being equal to $2m$. The radial modal index $q = 1, 2, \ldots$ will be used to label different resonances with the same azimuthal modal index m. We will discuss mathematical and physical interpretations of the radial modal index q in the next sections.

In general, to study resonant modes we firstly solve numerically eq. (5) and eq. (6) for several azimuthal modal indices m and for the fixed refractive index $n = 1.5$, using as initial guesses a fine grid in the complex wave number plane. Then we numerically continue the solutions for increasing and decreasing n.

TM modes. – Figures 2 and 3 show the behavior of TM resonances given by the solutions of eq. (5) for several azimuthal modal indices m under variation of the refractive index n. For a fixed n, like for $n = 1.5$ in figs. 2 and 3, one can clearly distinguish between the internal and external resonances as they are located in well-separated regions of the complex wave number plane: the internal resonances have much smaller imaginary parts in comparison to the external resonances. For each of the two kinds of resonances with the same m, we consecutively assign a radial modal index q in accordance with the increase of their real parts k_r starting from $q = 1$.

Another difference between internal and external resonances (in addition to their location in the complex wave number plane) is the number of radial modes in each group of fixed azimuthal index m. While there are infinitely many internal resonances for each azimuthal index $m \geqslant 0$, there are, as we will see below, only a finite number of external resonances for a given m, namely, none if m is 0 or 1, $\frac{m}{2}$ if m is even, and $\frac{m-1}{2}$ if m is odd. This is in agreement with the results of ref. [6] where the asymptotic formulas for TM external resonances are derived.

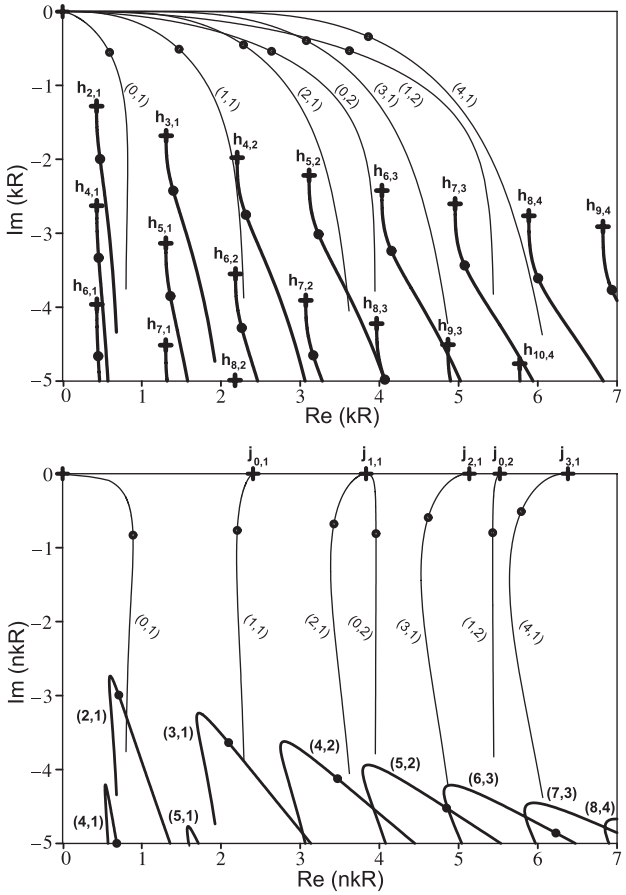

Fig. 2: TM internal (thin curves) and external (thick curves) resonances of a dielectric microdisk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the complex kR plane (upper panel) and nkR plane (lower panel). The filled circles correspond to $n = 1.5$.

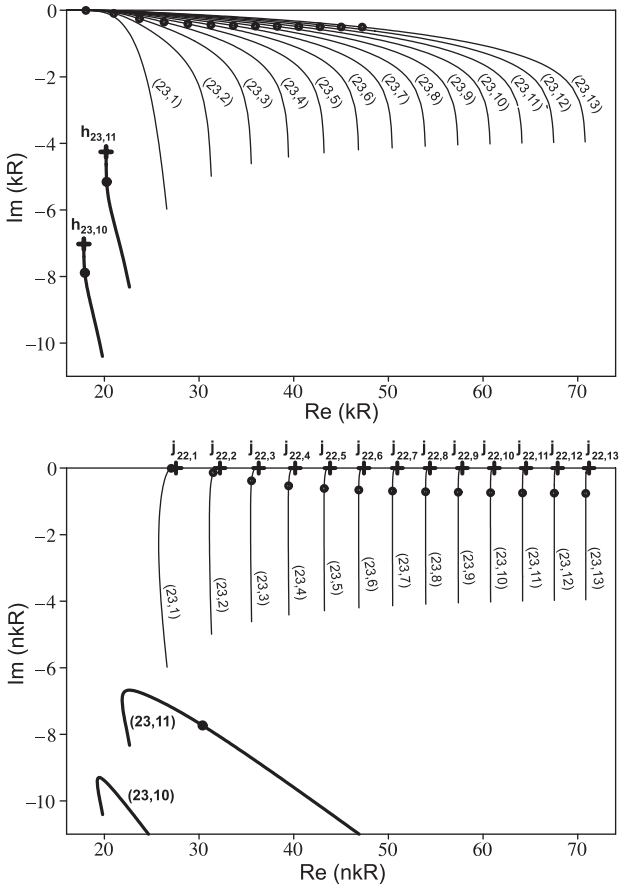

Fig. 3: TM internal (thin curves) and external (thick curves) resonances with the azimuthal modal index $m = 23$ of a dielectric microdisk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the regions $16 < \mathrm{Re}(kR) < 74$, $-11 < \mathrm{Im}(kR) < 0$ of the complex kR plane (upper panel) and nkR plane (lower panel). The filled circles correspond to $n = 1.5$.

For TM internal resonances, the physical meaning of the radial modal index q is the number of intensity hotspots in the radial direction inside of the disk, see fig. 4. For TM external resonances, the index q has no similar physical interpretation. These resonances are so deep in the complex wave number plane that the corresponding Bessel functions, see eq. (4), have almost no variation inside the disk. This is illustrated in fig. 5.

Let us now study the behavior of the scaled wave numbers $nk_{m,q}R$ of TM internal resonances in the small opening limit, $n \to \infty$. One would intuitively expect that in such a limit these resonances reduce to the real eigenvalues of a closed disk which are given by the corresponding zeros of Bessel functions $j_{m,q}$. However, as we will see below, this is not the case. In fact, for the TM internal resonances, see the thin curves in figs. 2 and 3, we have

$$\begin{aligned} \lim_{n\to\infty} nk_{m,q}R &= j_{m-1,q}, \quad m \neq 0, \\ \lim_{n\to\infty} nk_{0,q}R &= j_{1,q-1}, \quad q \neq 1, \\ \lim_{n\to\infty} nk_{0,1}R &= 0. \end{aligned} \tag{7}$$

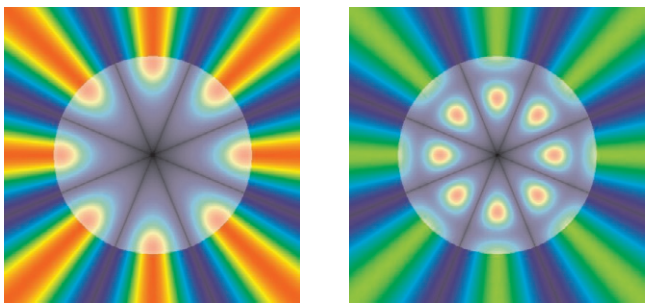

Fig. 4: (Color online). The intensity of TM internal resonant modes with the indicated modal indices in the near-field region of the dielectric disk with $n = 1.5$, $R = 1$. Red indicates high intensities, purple/black indicates low intensities.

This can be obtained by rewriting eq. (5) as

$$\frac{1}{n}\frac{J_m(knR)}{J_{m+1}(knR)} = \frac{H_m(kR)}{H_{m+1}(kR)}.$$

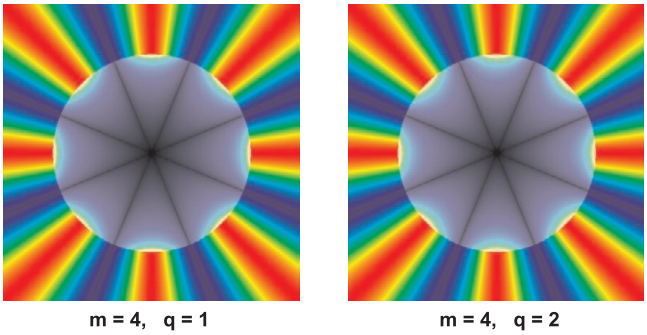

Fig. 5: (Color online). The intensity of TM external resonant modes with the indicated modal indices in the near-field region of the dielectric disk with $n = 1.5$, $R = 1$.

Then, from inspecting figs. 2 and 3 we see that the quantity nkR converges to finite real values and the quantity kR converges to 0 as $n \to \infty$ for all TM internal resonances. But for $kR \to 0$ we have

$$\frac{H_m(kR)}{H_{m+1}(kR)} \sim \begin{cases} kR/(2m), & m > 0, \\ (i\pi/2 - \ln(kR/2) - \gamma)\, kR, & m = 0, \end{cases}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. As a result, we obtain for $n \to \infty$ and $m \neq 0$,

$$\frac{1}{n}\frac{J_m(knR)}{J_{m+1}(knR)} \to \frac{kR}{2m},$$

or equivalently

$$J_{m-1}(knR) = \frac{2m}{knR} J_m(knR) - J_{m+1}(knR) \to 0,$$

i.e. all n scaled TM resonance wave numbers $nk_{m\neq 0,q}R$ approach the zeros $j_{m-1,\,q}$ (rather than $j_{m,\,q}$). Then, for $n \to \infty$ and $m = 0$ we have

$$\frac{knR\, J_1(knR)}{J_0(knR)} \sim \frac{1}{i\pi/2 - \ln(kR/2) - \gamma} \to 0.$$

Since J_0, J_1 are regular along the real axis, we have that $nk_{0,1}R \to 0$ and $nk_{0,q\neq 1}R \to j_{1,q-1}$ as illustrated in fig. 2.

As for TM external resonances, their wave numbers kR assume (finite) complex (not real) limits, and accordingly their scaled wave numbers, nkR, go to infinity. Using standard asymptotics for Bessel functions, one sees that

$$\frac{1}{n}\frac{J_m(knR)}{J_{m+1}(knR)} \to 0\,, \quad n \to \infty\,.$$

This immediately leads to

$$H_m(kR) \to 0, \qquad n \to \infty,$$

i.e. all TM external resonance wave numbers (not scaled with respect to n) satisfy the relation

$$\lim_{n\to\infty} k_{m,q}R = h_{m,q}, \tag{8}$$

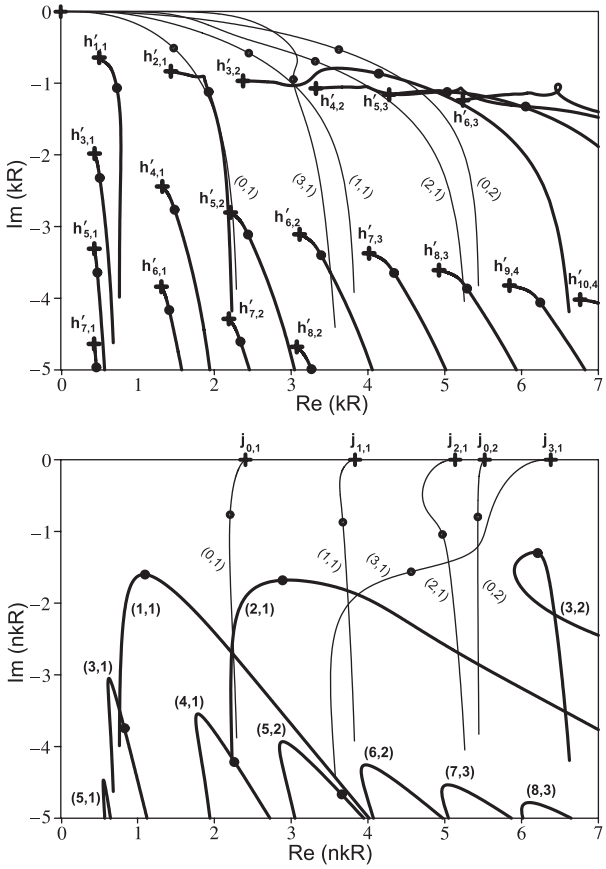

Fig. 6: TE internal (thin curves) and external (thick curves) resonances of a dielectric microdisk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the complex kR plane (upper panel) and nkR plane (lower panel). The filled circles correspond to $n = 1.5$.

where $h_{m,q}$ are complex zeros of Hankel functions. It is known, see ref. [10], that there is only a finite number of such zeros for a given m: 0 if m is 1 or 2, $m/2$ if m is even, and $(m-1)/2$ if m is odd. This exactly corresponds to our findings for the number of radial modes in each group of external resonances with fixed m.

TE modes. – Figures 6 and 7 show the behavior of TE resonances given by the solutions of eq. (6). Like for TM polarization, we separate internal and external resonances with the same m and assign the radial modal index q to each member of those two sets independently, in accordance with the increase of their real parts k_r starting from $q = 1$. But there could be a problem. The TE external resonances $k_{m,\frac{m}{2}}$ with $m = 2, 4, \ldots$ and $k_{m,\frac{m+1}{2}}$ with $m = 1, 3, \ldots$, *i.e.* the last ones in the sets of fixed m, do not necessarily have large imaginary parts. (We will call these resonances the "special" ones.) As a result, they could be mixed with TE internal resonances and their field intensities could display some features of internal resonances as well. Therefore, the only way to separate them is to trace them with increasing n till they reach

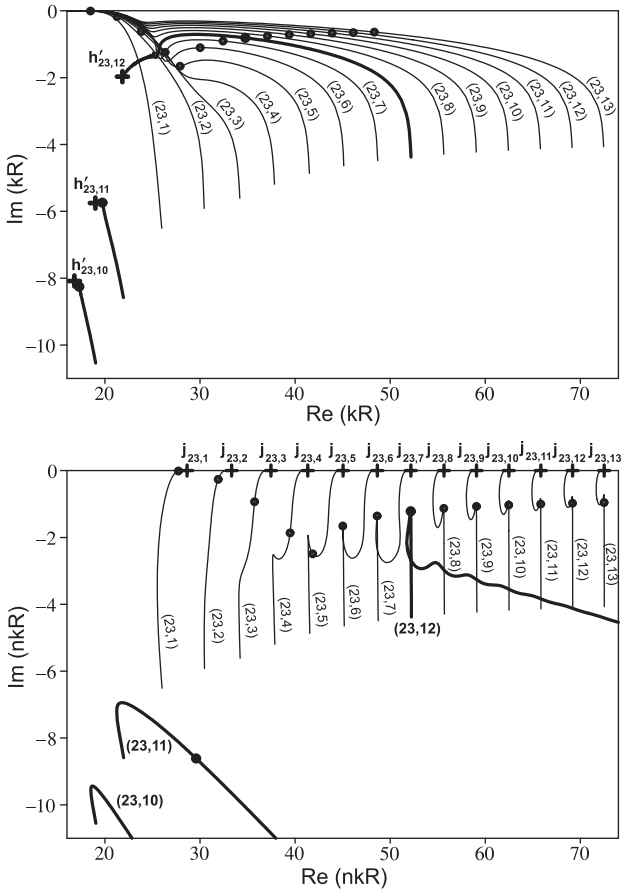

Fig. 7: TE internal (thin curves) and external (thick curves) resonances with the azimuthal modal index $m = 23$ of a dielectric microdisk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the regions $16 < \mathrm{Re}(kR) < 74$, $-11 < \mathrm{Im}(kR) < 0$ of the complex kR plane (upper panel) and nkR (lower panel). The filled circles correspond to $n = 1.5$.

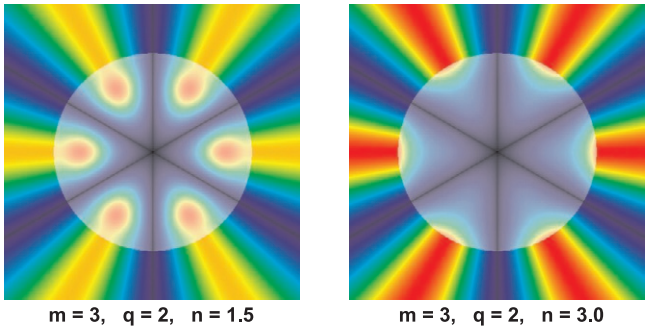

Fig. 8: (Color online). The intensity of TE external resonant modes with the indicated modal indices in the near-field region of the dielectric disk with $R = 1$ and $n = 1.5$ (left panel), $n = 3.0$ (right panel).

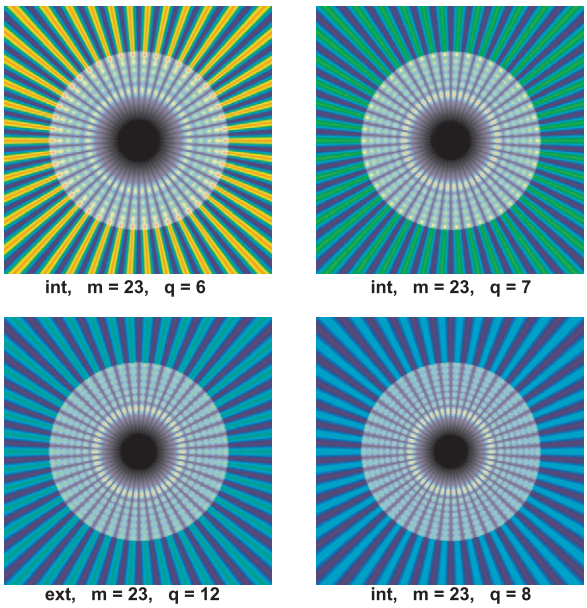

Fig. 9: (Color online). The intensity of TE resonant modes with the indicated modal indices in the near-field region of the dielectric disk with $R = 1$ and $n = 1.5$; "int" stands for internal resonance, "ext" stands for external resonance.

their limits (when $n \to \infty$), see the thick and thin curves in figs. 6 and 7.

Using arguments similar to the ones in the previous section on TM modes we find that eq. (6) takes the form $J_m(knR) = 0$ for internal and $H'_m(kR) = 0$ for external resonances when $n \to \infty$. This means that for the scaled wave numbers of the TE internal resonances

$$\lim_{n\to\infty} nk_{m,q}R = j_{m,q}, \tag{9}$$

as we intuitively expected. The TE external resonances (not scaled with respect to n) approach the complex zeros $h'_{m,q}$ of the corresponding Hankel function derivatives

$$\lim_{n\to\infty} k_{m,q}R = h'_{m,q}. \tag{10}$$

The thin and thick curves in figs. 6 and 7 illustrate the results numerically.

The field intensities of most TE modes display patterns similar to TM modes: the modal index q for internal resonances gives the number of intensity hotspots in the radial direction; for external resonances, which are deep in the complex wave number plane, there is almost no field variation inside the disk. However, the situation is different for the "special" TE external resonances. With the variation of the disk refractive index their imaginary parts could become relatively small. Then their field intensity patterns become similar to those of internal resonances, see the left panel in fig. 8. Moreover, for large azimuthal indices m and relatively low refractive indices n the "special" external resonances occupy positions exactly where one would expect the corresponding internal resonances, see the modes with $n = 1.5$ in fig. 7. As a result, the field patterns of internal resonances to the left from the "special" external one display some unexpected features as well, see fig. 9. For example, the field intensities of internal resonances $k_{23,6}$ and $k_{23,7}$ have five and six (rather than six and seven) intensity peaks in the radial direction.

Conclusions. – To summarize, we have studied in detail the behavior of both internal (Feshbach) and external (shape) resonances of an open microdisk in the small opening limit, *i.e.* when the microdisk refractive index goes to infinity, for both TM and TE polarizations. Contrary to naive expectations, the limit values of the open-disk resonances match the eigenvalues of the corresponding closed disk with the zero (Dirichlet) boundary conditions only for TE internal resonances. Based on the obtained limits, a clear separation of all resonant modes into internal and external ones has been achieved. While such a classification is of interest in its own right, it should also be useful for the construction of a trace formula for open cavities, see ref. [6]. Moreover, this classification is important for understanding the resonance level dynamics in circular cavities perturbed by a point scatterer, see ref. [11].

Furthermore, our analysis assigns mathematically unambiguous azimuthal and radial modal indices to each internal and external resonant mode. We showed that the latter index has a clear physical interpretation only for internal resonances, with one qualification. As the refractive index n is decreased, one observes the striking phenomenon that some special TE external resonances join the set of internal resonances and share their features. Our results should be of general interest since to the best of our knowledge this is the first complete classification of all resonant modes in the well-known and the simplest open system —a dielectric microdisk.

* * *

This work was supported by the EPSRC under grant No. EP/C515137/1.

REFERENCES

[1] Vahala K. J., *Nature*, **424** (2003) 839.
[2] Nosich A. I., Smotrova E. I., Boriskina S. V., Benson T. M. and Sewell P., *Opt. Quantum Electron.*, **39** (2007) 1253.
[3] Smotrova E. I., Nosich A. I., Benson T. M. and Sewell P., *IEEE J. Sel. Topics Quantum Electron.*, **11** (2005) 1135.
[4] Lebental M., Djellali N., Arnaud C., Lauret J.-S., Zyss J., Dubertrand R., Schmit C. and Bogomolny E., *Phys. Rev. A*, **76** (2007) 023830.
[5] Dubertrand R., Bogomolny E., Djellali N., Lebental M. and Schmit C., *Phys. Rev. A*, **77** (2008) 013804.
[6] Bogomolny E., Dubertrand R. and Schmit C., *Phys. Rev. E*, **78** (2008) 056202.
[7] Harge T. and Lebeau G., *Invent. Math.*, **118** (1994) 161.
[8] Sjostrand J. and Zworski M., *Ark. Mat.*, **33** (1995) 135.
[9] Ryu J.-W., Rim S., Park Y.-J., Kim C.-M. and Lee S.-Y., *Phys. Lett. A*, **372** (2008) 3531.
[10] Erdelyi A., *Higher Transcendential Functions*, Vol. **II** (McGraw-Hill, New York) 1953.
[11] Dettmann C. P., Morozov G. V., Sieber M. and Waalkens H., arXiv:0905.2351 (2009).