Statistical mechanics problem sheet 3

1. Does an ideal gas satisfy the third law of thermodynamics? Explain.

2. Give a rough estimate of the amount of molten salt required to store $10MWh$ of heat energy from a solar furnace (h is hour).

3. Calculate the pressure and density profile for an isothermal atmosphere of an ideal gas.

4. $*$ Consider an atmosphere containing two species with molecular masses m_1, m_2 and effective degrees of freedom d_1, d_2. Assume ideal conditions, ie no interactions between the particles.

   (a) Show that the equation $pV = NkT$ remains valid, where $N = N_1 + N_2$.

   (b) Express E in terms of the other quantities.

   (c) Find the corresponding adiabatic law

   (d) Describe the adiabatic atmosphere

5. $**$ Derive the Hamiltonian equations of motion (for those who have not seen them before), as follows:

   (a) We want a function $\mathbf{q}(t)$ which minimises (or in general is a stationary point of) the action (not entropy!)

   $$S = \int_{t1}^{t2} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt$$

   subject to fixed constraints $\mathbf{q}(t_1) = \mathbf{q}_1$, $\mathbf{q}(t_2) = \mathbf{q}_2$. L is called the Lagrangian function. If $\mathbf{q}(t)$ is perturbed by a function $\delta\mathbf{q}(t)$ which vanishes at the end points, show that the above integral is independent of $\delta\mathbf{q}$ to first order, and hence is a stationary value (possibly a minimum) if

   $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}}$$

   where the vector notation simply means that there is one equation for each component of $\mathbf{q}$.

   (b) Perform a Legendre transformation on each component to find a function $H(\mathbf{q}, \mathbf{p}, t)$ where $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$ and show that the above equations are equivalent to Hamilton's equations as defined in lectures.

6. $*$ A particle with charge e in an electromagnetic field has Hamiltonian

   $$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}) \cdot (\mathbf{p} - e\mathbf{A}) + e\Phi$$

   where $(\Phi, \mathbf{A})$ are electromagnetic potentials and functions of position $\mathbf{q}$, and all bold symbols are 3D vectors. Write down the equations of motion, and show that in this case, $\mathbf{p}$ (the "canonical" momentum) is not simply mass times velocity. Is this system time reversible?

7. For the 1D harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2)$$

Sketch the orbits in the phase plane (q, p) and give an explicit expression for the flow Φ^t and for the time evolution of a probability density with specified initial condition

$$\rho(q, p, 0) = \rho_0(q, p)$$

8. A common interatomic potential energy function is given by the Lennard-Jones function

$$U_{LJ}(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$$

Sketch this function, noting any critical points or intercepts. For the 1D motion in this potential

$$H = \frac{p^2}{2m} + U_{LJ}(q)$$

sketch the orbits in the phase plane. $*$ What would you expect to be the behaviour of many atoms interacting with this potential at high energy, and at low energy?