An Approximation of the First Order Marcum Q-Function with Application to Network Connectivity Analysis

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Abstract—An exponential-type approximation of the first order Marcum Q-function is presented, which is robust to changes in its first argument and can easily be integrated with respect to the second argument. Such characteristics are particularly useful in network connectivity analysis. The proposed approximation is exact in the limit of small first argument of the Marcum Q-function, in which case the optimal parameters can be obtained analytically. For larger values of the first argument, an optimization problem is solved, and the parameters can be accurately represented using regression analysis. Numerical results indicate that the proposed methods result in approximations very close to the actual Marcum Q-function for small and moderate values of the first argument. We demonstrate the accuracy of the approximation by using it to analyze the connectivity properties of random ad hoc networks operating in a Rician fading environment.

Index Terms—Marcum Q-function, approximation, network connectivity.

I. INTRODUCTION

THE Marcum Q-function, defined as the integral [1]

$$Q_1(a,b) = \int_b^\infty x \exp\left(-\frac{x^2 + a^2}{2}\right) I_0(ax)\mathrm{d}x \qquad (1)$$

for $a, b \geq 0$ where $I_0(x)$ is the modified Bessel function of the first kind, is a fundamental function that arises in the performance evaluation of a wide class of communication systems [1]–[3]. From a mathematical point of view, this function represents the complementary cumulative distribution function (CCDF) of the power of a Rician distribution. The integral representation of the function given by (1) cannot be manipulated easily to provide simple expressions for the performance of communication systems, especially when the function $Q_1(a,b)$ must be integrated with respect to one of its arguments [1]. To solve this issue, numerous works have proposed alternative representations of $Q_1(a,b)$ to facilitate analysis (see, e.g., [3], [4] and references therein). Exponential-type bounds, provided they are tight, have been particularly attractive, especially when evaluating the bit error rate at high signal to noise ratio (SNR) [3], [5]. In other situations, approximations may be more suitable than bounds [6], [7]. However, such approximations may have complicated

mathematical structures and/or be inaccurate in certain domains of their arguments.

In this paper, a simple exponential approximation of the first order Marcum Q-function is presented that yields small approximation error over a large domain in its two arguments. The approximation is designed such that it can be used in situations where $Q_1(a,b)$ must be integrated over its second argument. In what follows, a heuristic approach is first employed to find the right form of the approximation, which is parameterized by two functions of a. An analytical framework for determining the correct parameterization is then explored, which is shown to be accurate for $0 \leq a \ll 1$. For $a \gg 1$, the optimal parameterization is calculated numerically. Although the proposed approximation is useful in its own right, it is particularly helpful in situations where the Q-function must be integrated. To illustrate this fact, we present an example application of our results whereby the proposed approximation is used to analyze the connectivity probability of a random ad hoc network operating in Rician fading channels.

The structure of this paper is as follows. In Section II, the proposed approximation and means of deriving the optimal a-dependent parameters are presented. An example application of the approximation is given in Section III, while the accuracy of the approximation is presented in Section IV. Finally, some concluding remarks are given in Section V.

II. APPROXIMATION OF $Q_1(a,b)$

By plotting $Q_1(a,b)$ as a function of b for various values of a, it can be readily observed that $Q_1(a,b)$ decays exponentially with b, where the value of a roughly defines the shift of Q_1 along the b-axis. Consequently, we propose to approximate $Q_1(a,b)$ by the function

$$\tilde{Q}_1(a,b) = \exp\left(-e^{\nu(a)}b^{\mu(a)}\right) \qquad (2)$$

where $\nu(a)$ and $\mu(a)$ are nonnegative parameters dependent upon a. The key is to choose these parameters such that the accuracy of the approximation is high.

As previously discussed, we are concerned with obtaining an approximation that is useful over the range of the argument b for some fixed a. Thus, we define the approximation error as the function

$$\mathcal{E}(a) = \int_0^\infty (Q_1(a,b) - \tilde{Q}_1(a,b))^2 \mathrm{d}b. \qquad (3)$$

Furthermore, we define $\mu(a)$ and $\nu(a)$ to be polynomials of order m, where a larger value of m yields a better approximation. Thus, we have

$$\mu(a) = \mu_0 + \mu_1 a + \mu_2 a^2 + \cdots + \mu_m a^m$$
$$\nu(a) = \nu_0 + \nu_1 a + \nu_2 a^2 + \cdots + \nu_m a^m$$

in which case the approximation becomes

$$\tilde{Q}_1(a,b) = \exp\left(-e^{\sum_{n=0}^{m}(\mu_n \ln b + \nu_n)a^n}\right). \qquad (4)$$

The goal is now to choose the coefficients $\{\mu_0, \ldots, \mu_m, \nu_0, \ldots, \nu_m\}$, independent of b, such that $\mathcal{E}(a)$ is minimized. Depending on the value of the argument a, this can be done analytically or numerically.

A. Analytical Approach for Small Arguments

First, consider the case where $0 \le a \ll 1$. It is logical to expand $Q_1(a,b)$ and $\tilde{Q}_1(a,b)$ about $a = 0$ and equate the coefficients term by term. Of course, if the expansions for Q and $\tilde{Q}$ converge and the corresponding coefficients match to arbitrary order, then $Q = \tilde{Q}$. Since this is clearly not the case, it is advisable to equate coefficients recursively, from lowest to highest order.

For example, let $m = 4$. To leading order, we have $Q_1(0,b) = \exp(-b^2/2)$ and $\tilde{Q}_1(0,b) = \exp(-e^{\nu_0}b^{\mu_0})$. It follows that we should choose $\mu_0 = 2$ and $\nu_0 = -\ln 2$ since this ensures the approximation is exact at $a = 0$. Next, we can equate the first order terms to obtain the equation[1] $\mu_1 \ln b + \nu_1 = 0$. But we see from (4) that this formula implies there is no $O(a)$ term in the second exponent of $\tilde{Q}$. Thus, we may take $\mu_1 = \nu_1 = 0$ to maintain independence of b. This process can be continued in a straightforward manner. However, when we equate the fourth order terms, we obtain the equation $\mu_4 \ln b + \nu_4 = b^2/32$, and thus either μ_4 or ν_4 is dependent upon b, a condition that is not allowed by our definition of the polynomials μ and ν. Instead, we can optimize $\mathcal{E}(a)$ over μ_4 and ν_4 by differentiating with respect to each variable, setting the results to zero, and solving for μ_4 and ν_4. This yields the optimal fourth order polynomials

$$\mu(a) = 2 + \frac{9}{8(9\pi^2 - 80)}a^4$$

$$\nu(a) = -\ln 2 - \frac{a^2}{2} + \frac{45\pi^2 + 72\ln 2 + 36C - 496}{64(9\pi^2 - 80)}a^4$$

which are independent of b, and thus satisfy the conditions of our approximation[2]. By substituting these expressions for $\mu(a)$ and $\nu(a)$ into $\tilde{Q}$ and evaluating the integral in (3) for small a, it is apparent that $\mathcal{E}(a) \approx 7.5 \times 10^{-5}a^8$. Thus, the fourth order result is very accurate for $a \ll 1$, an observation that is corroborated by Fig. 1.

B. Numerical Approach for General Arguments

While a closed-form expression for the coefficients of μ and ν can be obtained for small a, performing a similar analysis for larger values of a is somewhat problematic. On that account, a numerical approach is followed instead. In particular, we propose to determine the appropriate values of μ and ν such that the following error is minimized:

$$\hat{\mathcal{E}} = \delta \sum_{\beta=0}^{\infty} (Q_1(a, \delta\beta) - \exp(-e^{\nu}(\delta\beta)^{\mu}))^2 \qquad (5)$$

[1]The details of the calculations are straightforward but lengthy, and are thus omitted here for brevity.

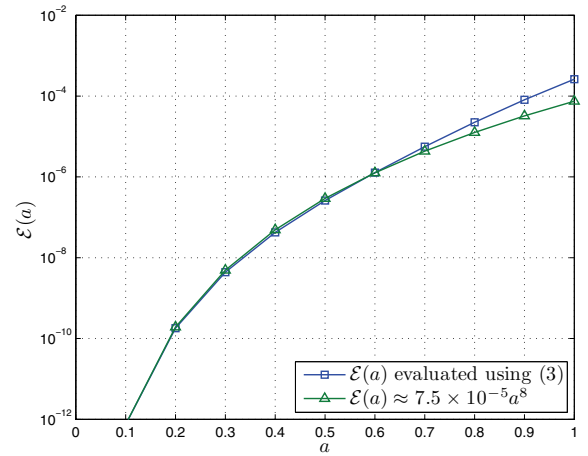[2]The symbol C denotes the Euler-Mascheroni constant, where $C \approx 0.5772$.

Fig. 1. Approximation error derived from fourth order polynomial representation of $\mu(a)$ and $\nu(a)$ for small a.

TABLE I
SUITABLE ν AND μ FOR DIFFERENT a

a	ν	μ
1.0000	-1.1739	2.0921
2.0000	-2.5492	2.7094
3.0000	-4.6291	3.6888
4.0000	-7.1668	4.7779
5.0000	-10.0339	5.9074
6.0000	-13.2014	7.0794

where δ is small and $\hat{\mathcal{E}} \to \mathcal{E}$ as $\delta \to 0$. Such optimization problems can be solved using numerical techniques [8]. It should be noted that the problem of minimizing the error term defined in (5) is not a convex optimization problem. As such, numerical methods may not always converge to the global optimum. Nevertheless, we find that the observed optimum is often adequate, as illustrated in Section IV.

Since $Q_1(a,b)$ decays exponentially with b, we argue that we can ignore terms in the summation in (5) corresponding to values of b larger than some $b_{\max}$ in order to facilitate optimization. This is particularly justified by noting that we are interested in obtaining an accurate expression for Q_1 that captures most of its mass. Thus, the upper limit on the summation in (5) can be replaced by $\beta_{\max} = b_{\max}/\delta$. As an example, we set $\delta = 10^{-4}$ and $b_{\max} = 12$, and solved the above optimization problem using a line-search algorithm for several values of a. Results are shown in Table I.

Using the values listed in Table I, it is possible to derive an approximate expression for $\mu(a)$ and $\nu(a)$ using polynomial regression [9]. For instance, assuming that $\mu(a)$ is a polynomial of fourth order in a, the regression model for $\mu(a)$ can be expressed as

$$\mu(a_j) = \tilde{\mu}_0 + \tilde{\mu}_1 a_j + \tilde{\mu}_2 a_j^2 + \tilde{\mu}_3 a_j^3 + \tilde{\mu}_4 a_j^4 + \epsilon_j \qquad (6)$$

for $j = 1, \ldots, N$ where ϵ_j is the error in the approximation, $\{\tilde{\mu}_i\}_{i=1}^4$ are the estimation coefficients, and N is the number of observed instances (c.f. Table I). The above expression can be written in matrix form as $\boldsymbol{\mu} = \mathbf{A}\tilde{\boldsymbol{\mu}} + \boldsymbol{\epsilon}$, where $\tilde{\boldsymbol{\mu}} = [\tilde{\mu}_0, \cdots, \tilde{\mu}_4]^T$ and $\boldsymbol{\mu}$ is similarly defined, $\mathbf{A}$ is an $N \times 5$ matrix with kth column being $[a_1^{k-1}, \cdots, a_N^{k-1}]^T$ for $k = 1, \ldots, 5$, and $\boldsymbol{\epsilon} = [\epsilon_1, \cdots, \tilde{\epsilon}_N]^T$. Using ordinary
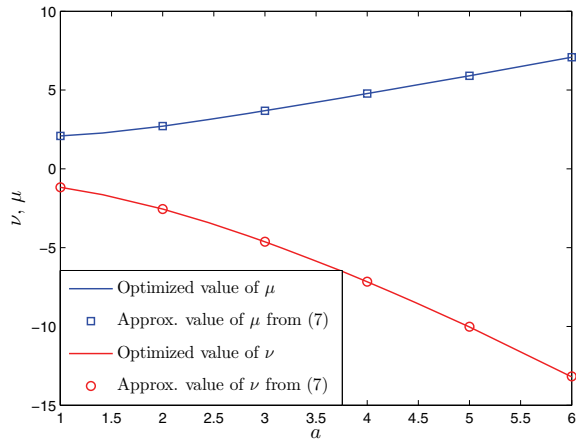
Fig. 2. Plots of $\mu(a)$ and $\nu(a)$ obtained through a numerical approach. The solid lines represent the optimized values, while the squares and circles show the approximate values from (7).

least squares estimation, the coefficients can be obtained using $\tilde{\boldsymbol{\mu}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\boldsymbol{\mu}$. For $m = 4$, this approach yields

$$\mu(a) = 2.174 - 0.592a + 0.593a^2 - 0.092a^3 + 0.005a^4$$
$$\nu(a) = -0.840 + 0.327a - 0.740a^2 + 0.083a^3 - 0.004a^4. \tag{7}$$

Fig. 2 depicts the comparison between the optimized values (from Table I) and approximated values of the two parameters given in (7). It can be observed from the plots that the two sets of values are very close, indicating the suitability of the above two equations. Such approximations are convenient if fast computation of the parameters $\nu(a)$ and $\mu(a)$ are required.

III. AN APPLICATION OF THE PROPOSED APPROXIMATION

As previously stated, the presented approximation can be particularly useful when the CCDF of the power of a Rician channel needs to be integrated over the second argument. The power of a Rician channel is noncentral-χ^2 distributed, whose CCDF is given by

$$F_X(x) = Q_1\left(\sqrt{2K}, \sqrt{2\omega^{-1}(K+1)x}\right) \tag{8}$$

where K is the Rice factor and ω is a channel dependent parameter. Given that $a = \sqrt{2K}$ in this case and, in general, $1 \leq K \leq 10$ [10], it follows that $a < 5$. On that account, the approximations of the parameters $\mu(a)$ and $\nu(a)$ presented above can readily be used.

To demonstrate the use of the proposed approximation of the Marcum Q-function, we consider the analysis of the full connection probability of a random ad hoc network, similar to the work presented in [11], [12]. Consider the connection probability of two nodes in a system, which we denote by H, given a minimum data rate requirement of R_0. By adopting an information theoretic definition of connectivity, we define

$$H = P\left(\log_2(1 + \gamma|h|^2) \geq R_0\right) \tag{9}$$

where $|h|^2$ is the channel gain between the two nodes and γ is the SNR which is dependent upon the distance between the two nodes and other parameters such as the path loss

exponent and antenna gains. Under the assumption of a Rician fading channel with Rice factor K and a path loss exponent of two for illustration, we have $H(r) = Q_1\left(\sqrt{2K}, r\alpha\right)$ where r is the distance between the two nodes and α is a function of the system parameters. To derive the probability that the network is fully connected, it is necessary to average $H(r)$ over the configuration space [12]. For a homogeneous system, this amounts to averaging $H(r)$ over all distances between nodes. Such a calculation would involve an integral of the form (c.f., (19)-(21) in [12] for Rayleigh fading)

$$\int_{r_1}^{r_2} rH(r)dr = \int_{r_1}^{r_2} rQ_1\left(\sqrt{2K}, r\alpha\right)dr$$
$$\approx \int_{r_1}^{r_2} re^{-e^{\nu}(r\alpha)^{\mu}}dr$$
$$= \frac{1}{\mu}\lambda^{-\frac{2}{\mu}}\left(\gamma(\frac{2}{\mu}, \lambda r_2^{\mu}) - \gamma(\frac{2}{\mu}, \lambda r_1^{\mu})\right) \tag{10}$$

where $\lambda = e^{\nu}\alpha^{\mu}$, r_1 and r_2 are the minimum and maximum distances between nodes within the system, and $\gamma(x,y)$ is the lower incomplete gamma function. The complete analysis of the full connection probability is beyond the scope of this letter. What is important to note is that without the approximation derived in this paper, solving the integral stated above would be very challenging if not impossible.

IV. ACCURACY OF THE APPROXIMATION

To demonstrate the accuracy of the proposed approximation, we compare the proposed method to existing approximations in the literature [7], [13] that generally yield small approximation errors. The comparisons are shown in Fig. 3. For the approximation presented in [7], the value of k was set to 50 in equation (6) therein. On the other hand, the approximation in [13] is obtained by taking the average of the lower and upper bounds of the $Q-$function as presented by the authors. It can be observed from the plot that, for small a, the approximations are close to the Marcum Q-function. However, as a increases, divergence from the actual curve is seen for the approximation of [7]. Nevertheless, the proposed approximation still adequately represents the mass of the Marcum Q-function over the range of b values; consequently, our approximation is robust with respect to changes in a, similar to [13]. It should be noted that, although the integral of the approximation in [13] is possible, the resulting mathematical expressions are considerably more complicated than the proposed method and thus do not easily lend themselves to further manipulations and calculations.

For large values of b, we note that existing bounds and approximations in the literature often provide a more accurate representation of $Q_1(a,b)$ compared to the proposed method. This can easily be observed graphically, but we omit the results here due to space constraints. We would also like to point out that some existing approximations [14] lead to very accurate representations of $Q_1(a,b)$ for all a and small values of b. As b increases however, such approximations diverge.

For applications that require a closer approximation for large b, the expressions in [4], [7] and references therein would be more appropriate. However, if the integral of the Marcum
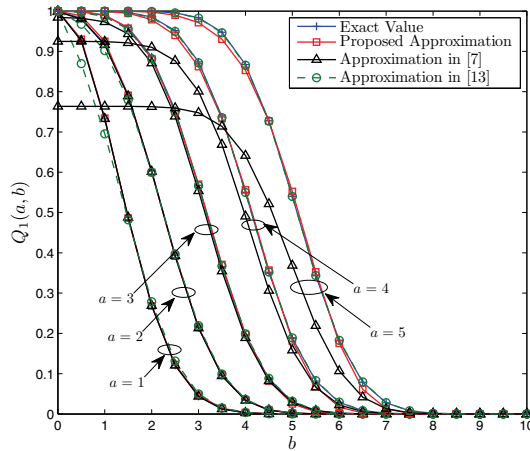
Fig. 3.   Comparison of the Marcum Q-function with the proposed approximation and that presented in [7] and [13]. For larger a, the approximation in [7] diverges from the actual curve for small b. Parameters for the proposed approximations are obtained from (7).
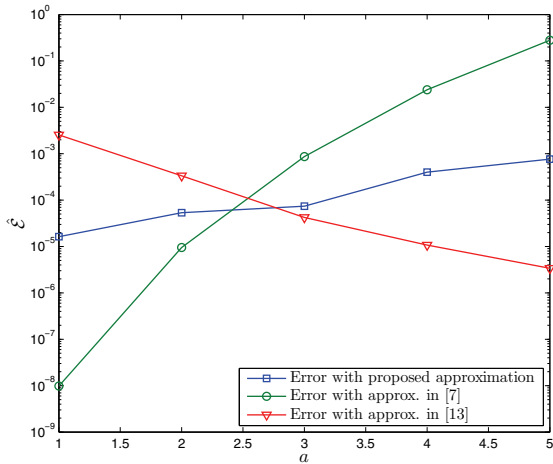

Fig. 4.   Comparison of the approximation error $\hat{\varepsilon}$ using the proposed approach and the ones given in [7] and [13]. The proposed approximation remains robust to changes in a.

Q-function over the domain of the second argument is sought, the approximation presented in this paper is more suitable.

We next compare $\hat{\varepsilon}$ for our proposed approximation and the one given in [7], [13]. Results shown in Fig. 4 for different values of a demonstrate the accuracy of the proposed scheme. As mentioned in the previous section, the range of a values considered in the plot is the range that would typically be encountered in practice in communication system analysis with Rician fading [10]. However, for the problem defined in (5), it is guaranteed that the solution would minimize the error term for any value of a.

V. Conclusion

In this paper, a simple approximation of the first order Marcum Q-function was presented that can be used in network connectivity analysis. For small input argument a, an analytical approach was presented for finding the approximation parameters, while for larger a, a numerical procedure based on an optimization problem was proposed. Equations for approximating these parameters were then presented. Simulation results demonstrated that the approximations led to an accurate representation of the Marcum Q-function, especially for small values of b. As b tends to infinity however, existing bounds of $Q_1(a, b)$ yield to a closer representation of the function.

Acknowledgement

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