Hypothesis Testing

Daniel Lawson

Wellcome Trust Sir Henry Dale Research Fellow
IEU, University of Bristol
Visiting researcher at the University of Oxford

October 2014
Gently adapted from David Steinsaltz's 2013 'hypothesis testing' slides

Table of Contents

Section 1

Background

The testing paradigm

Significance testing is about rejecting a null model.

- We have a research hypothesis, which helps define the alternative hypothesis
- The null model is our best explanation of the data without that hypothesis
- We see if the null model 'fits' the data with a test
- We will 'test' all of the assumptions of the null together!
- i.e. we won't know which assumption failed
- If we reject the null, we hope our alternative hypothesis is the explanation!
- Confounding by some unaccounted process is the most common reason for incorrectly accepted alternatives.
- If you can't account for all reasonable confounders, there is no point doing the test!

Decisions and hypothesis testing

One reason for HT is to make decisions. It is biased against the alternative, so that if the null is rejected we are fairly certain that is not a mistake.

- Example: A new vaccine.
- Null hypothesis: it doesn't work.
- Alternative: it does. So use it!

It is important to control the probability that we decide it works!
(But we aren't taking into account the *cost* of getting it wrong.)

Types of error

- Type I: We reject the null hypothesis, but its really true
- Type II: We retain the null, but it isn't true

These are very different!
In many fields, no null hypothesis can be true.
This leads to the question: do we have *power* to reject the null?
In this case, standard hypothesis testing can be useless for demonstrating a specific alternative.

The Confusion Matrix

		What we do?	
		Reject Null	Retain null
What is true?	null	False positive Type I error	True negative
	alternative	True positive	False negative Type II error

Power $= \frac{\text{true positive rate}}{\text{false negative rate}}$ for a given false positive rate α.

Example: Spam Filter

Spam filters look at incoming email, and use statistical models to compute the probability that a real message would have certain features. If the probability is too low, it goes directly in the bin.

Null hypothesis: Real message.
Alternative: Spam.

Then a Type I error is

1. A spam message that gets through the filter;
2. An email from your friend that gets junked?

Example: Spam Filter

Spam filters look at incoming email, and use statistical models to compute the probability that a real message would have certain features. If the probability is too low, it goes directly in the bin.

Null hypothesis: Real message.
Alternative: Spam.

Then a Type I error is

1. A spam message that gets through the filter;
2. An email from your friend that gets junked?

Type I = falsely rejecting the null

Example: Spam Filter

Spam filters look at incoming email, and use statistical models to compute the probability that a real message would have certain features. If the probability is too low, it goes directly in the bin.

Null hypothesis: Real message.
Alternative: Spam.

Then a Type I error is

1. A spam message that gets through the filter;
2. An email from your friend that gets junked?

Type I = falsely rejecting the null
Answer: 2

Example: Criminal Trial

Juries weigh the evidence, and decide how likely the evidence would be if the defendant were innocent.
Null hypothesis: Defendant is innocent.
Alternative: Defendant is guilty.

Then a Type II error is

1. A guilty defendant set free;
2. An innocent defendant convicted.

Example: Criminal Trial

Juries weigh the evidence, and decide how likely the evidence would be if the defendant were innocent.
Null hypothesis: Defendant is innocent.
Alternative: Defendant is guilty.

Then a Type II error is

1. A guilty defendant set free;
2. An innocent defendant convicted.

Type II = falsely rejecting the alternative

Example: Criminal Trial

Juries weigh the evidence, and decide how likely the evidence would be if the defendant were innocent.
Null hypothesis: Defendant is innocent.
Alternative: Defendant is guilty.

Then a Type II error is

1. A guilty defendant set free;
2. An innocent defendant convicted.

Type II $=$ falsely rejecting the alternative
Answer: 1

Example: Clinical Trial

A new cancer medication is tested in comparison to an old one. We test how likely the apparent improvement in survival would be, if the new drug were no better than the old one. The medication will be approved if its proved to be better.

Null hypothesis: The new medication is no better than the old one.
Alternative: The new medication is better.
Then a Type II error is

1. When we approve the new medication even though its no better than the old one;
2. When we dont approve the new medication, even though it is better.

Example: Clinical Trial

A new cancer medication is tested in comparison to an old one. We test how likely the apparent improvement in survival would be, if the new drug were no better than the old one. The medication will be approved if its proved to be better.

Null hypothesis: The new medication is no better than the old one.
Alternative: The new medication is better.
Then a Type II error is

1. When we approve the new medication even though its no better than the old one;
2. When we dont approve the new medication, even though it is better.

Answer: 2

Hypothesis testing

- Start with a 'research hypothesis' - the 'alternative hypothesis'
- Form a 'null hypothesis' that says nothing interesting happened – it was all chance variation
- Determine the test statistic T, measuring how unusual the data is under the null hypothesis
- Define a 'significance level' α
- Determine the critical value $T_{crit}(\alpha)$
- Compute test statistic T for the observed data
- Compute the p-value: $p(T)$ probability of a test statistic as extreme or more than that observed (under the null)
- Retain the null if $p > \alpha$, or equivalently $T < T_{crit}$. Otherwise reject it in favour of the alternative.

Section 2

The Z test

Subsection 1

The simple Z test

The simple Z test

For n data points X_i

- If the mean of the data can be treated as Normal,
- And our null hypothesis is $\overline{X} = \mu$ for known μ...
- And we know the standard deviation σ...
- Then we compute the test statistic $Z = (\overline{X} - \mu)/\sigma$...
- Under the null, $Z \sim N(0, 1)$

So the Z test computes

- $p(Z' \geq Z)$ (one tailed)
- $p(|Z'| \geq |Z|)$ (two tailed)
- where Z' is from the null

Important properties of the Normal distribution

If X is normal then $aX + b$ is normal
If X and Y are normal and independent, then $X + Y$ is normal.
Specifically, $Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Important properties of the Normal distribution

If X is normal then $aX + b$ is normal
If X and Y are normal and independent, then $X + Y$ is normal.
Specifically, $Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Meaning that we add the variance, regardless of whether we add X and Y!

Important properties of the Normal distribution

If X is normal then $aX + b$ is normal
If X and Y are normal and independent, then $X + Y$ is normal.
Specifically, $Z = X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Meaning that we add the variance, regardless of whether we add X and Y!
More generally, if X and Y are correlated,

$$\mathrm{var}(X + Y) = \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y$$

Law of large numbers

Let $X_1, \ldots, X_n$ be independent samples from a distribution with mean μ and variance σ^2. Then

$$\overline{X} := \frac{1}{n}(X_1 + \ldots + X_n)$$

converges to μ as $n \to \infty$.

- 'Estimate' of μ: $\overline{X}$
- Variance of the estimator: $\mathrm{var}(\overline{X}) = \sigma^2/n$
- 'standard error' SE: $\mathrm{SD}(\overline{X}) = \sigma/\sqrt{n}$

So $Z := \frac{\overline{X}-\mu}{\sigma/\sqrt{n}}$ is 'standardized' to have mean 0 and s.d. 1.

Takeaway message:

- You can assume that the mean of[1] a distribution is normal
- ... if n (sample size) is big enough
- IT DOESNT MATTER if the data you sampled were normally distributed.
- The mean still has to be normally distributed
- How big is 'big enough'? It depends on the distribution!

[1]: There are some special distributions that don't obey the law of large numbers. These have infinite variance.

Z test example

Time (min)	Sex	Weight (g)
5	F	3837
64	F	3334
78	M	3554
115	M	3838
177	M	3625
245	F	2208
247	F	1745
262	M	2846
271	M	3166
428	M	3520
455	M	3380
492	M	3294
494	F	2576
549	F	3208
635	M	3521
649	F	3746
653	F	3523
693	M	2902
729	M	2635
776	M	3920
785	M	3690
846	F	3430

Time (min)	Sex	Weight (g)
847	F	3480
873	F	3116
886	F	3428
914	M	3783
991	M	3345
1017	M	3034
1062	F	2184
1087	M	3300
1105	F	2383
1134	M	3428
1149	M	4162
1187	M	3630
1189	M	3406
1191	M	3402
1210	F	3500
1237	M	3736
1251	M	3370
1264	M	2121
1283	M	3150
1337	F	3866
1407	F	3542
1435	F	3278

- British birthweights have Mean 3426g and SD 538g from a large sample
- Australian birthweights sampled (1 day)
- Australians have mean 3276
- Null: birthweights have the same mean
- Alternative: Australian babies are smaller

Z test example

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- Let $X_1, \cdots, X_{44}$ be the birth weights in the sample
- If the null hypothesis holds,

$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, 538^2/44)$$

- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed $Z = (\overline{X} - 3426)/81 \sim N(0,1) = -1.81$

$p(Z \leq -1.81) = 0.035$ can be looked up in a table!
or computed using:

- Matlab: cdf('normal',-1.81)

Z test example

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- Let $X_1, \cdots, X_{44}$ be the birth weights in the sample
- If the null hypothesis holds,

$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, 538^2/44)$$

- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed
  $Z = (\overline{X} - 3426)/81 \sim N(0,1) = -1.81$

$p(Z \leq -1.81) = 0.035$ can be looked up in a table!
or computed using:

- Matlab: cdf('normal',-1.81)
- R: pnorm(-1.81)

Z test example

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- Let $X_1, \cdots, X_{44}$ be the birth weights in the sample
- If the null hypothesis holds,

$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, 538^2/44)$$

- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed
  $Z = (\overline{X} - 3426)/81 \sim N(0, 1) = -1.81$

$p(Z \leq -1.81) = 0.035$ can be looked up in a table!
or computed using:

- Matlab: cdf('normal',-1.81)
- R: pnorm(-1.81)
- Excel:

Z test example

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- Let $X_1, \cdots, X_{44}$ be the birth weights in the sample
- If the null hypothesis holds,

$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, 538^2/44)$$

- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed $Z = (\overline{X} - 3426)/81 \sim N(0, 1) = -1.81$

$p(Z \leq -1.81) = 0.035$ can be looked up in a table!
or computed using:

- Matlab: cdf('normal',-1.81)
- R: pnorm(-1.81)
- Excel: (I'm not going to encourage this!)

Z test example

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- Let $X_1, \cdots, X_{44}$ be the birth weights in the sample
- If the null hypothesis holds,

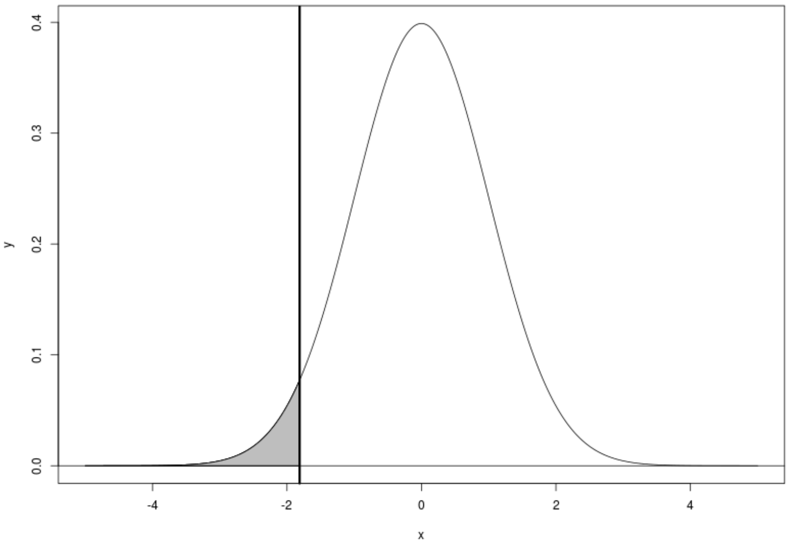
$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, 538^2/44)$$

- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed $Z = (\overline{X} - 3426)/81 \sim N(0,1) = -1.81$

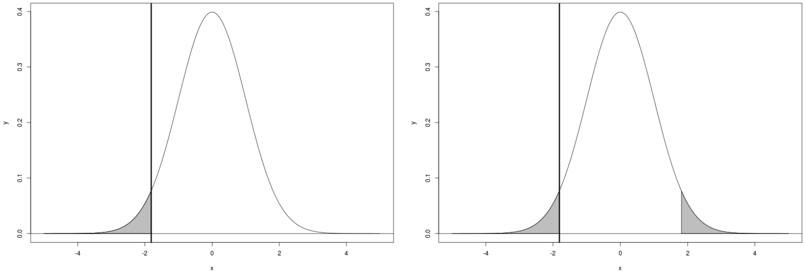
$p(Z \leq -1.81) = 0.035$ can be looked up in a table!
or computed using:

- Matlab: cdf('normal',-1.81)
- R: pnorm(-1.81)
- Excel: (I'm not going to encourage this!)
- Wolfram Alpha, your phone, Google, etc!

Tails of the Normal Distribution



Two tailed test

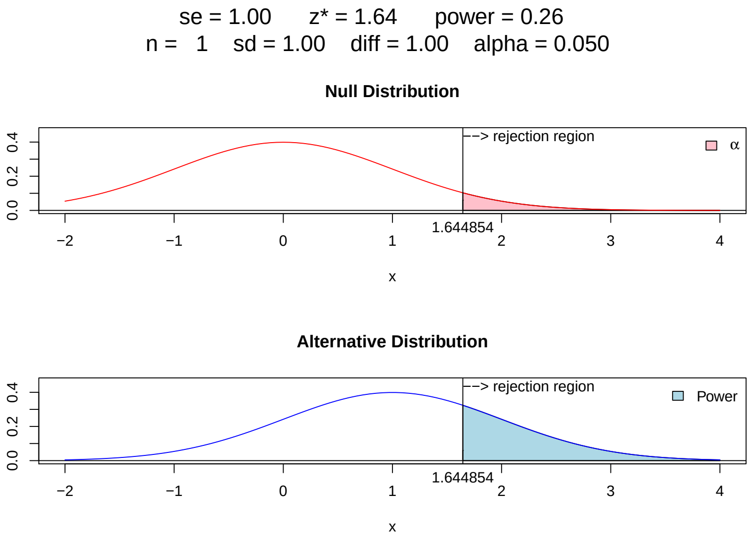


- P-value 0.035 in one-tailed test
- P-value 0.070 in two-tailed test
- For symmetric distributions, p-value always doubles

Does it matter how many tails?

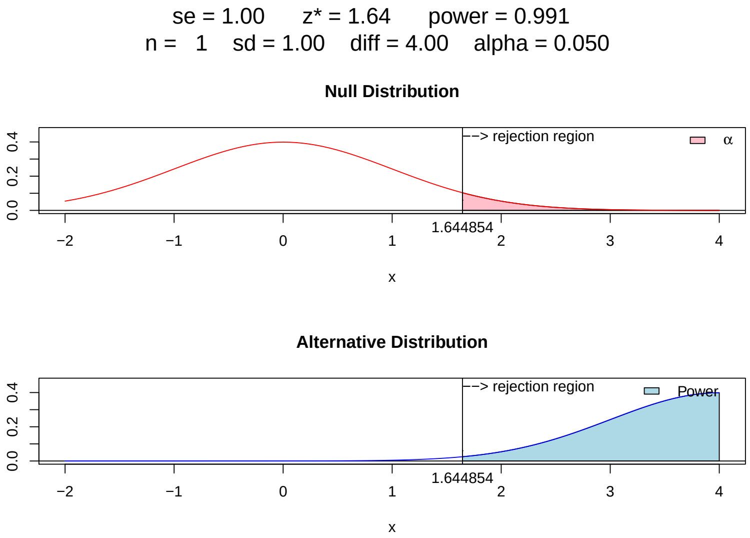
- Not really... Just make sure you're clear on why.
- Switching from two-tailed to one-tailed can make a non-significant result significant.
- Hypothesis testing is in the business of being conservative...
- You should therefore have to justify a one-tailed choice.
- Previously, before we looked at the data, did we expect Australian babies to be smaller? Would we not have been interested if they were bigger?
- If that was interesting too, we need a two-tailed test.

Requiem on errors: Power and alternative hypotheses



Power with 1 sd difference in the alternative

Requiem on errors: Power and alternative hypotheses



Power with 4 sd difference in the alternative

Subsection 2

The Z test for proportions

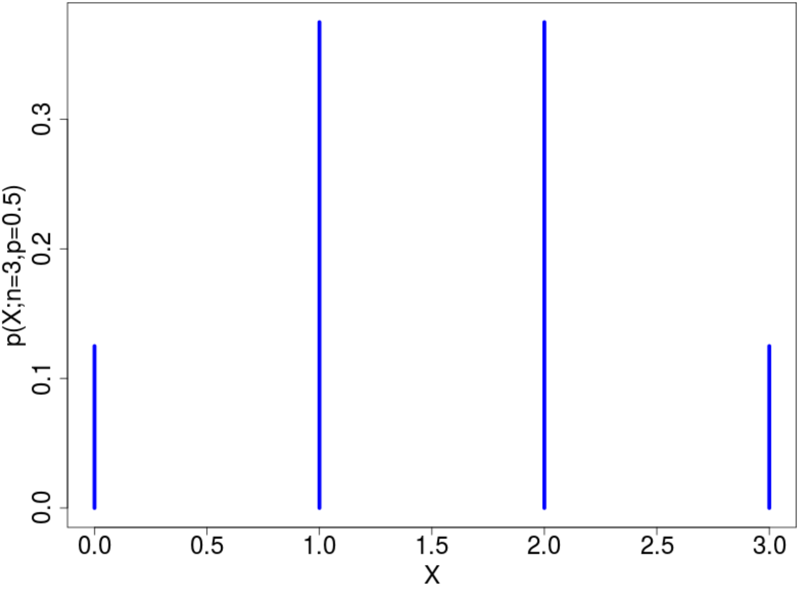
Testing the number of successes

- Test: observe X successes from n trials
- H0: $X/n = p$, we are testing the probability of a success
- Let $C_i \sim Bern(p)$
- i.e. Bernoilli with p=success probability,
- $X = \sum_{i=1}^{n} C_i \sim Binomial(n, p)$ is the number of successes
- Can compute $p(X \leq x; n, p)$ (too few successes) and $p(X \geq x; n, p)$ (too many successes) explicitly for moderate n
- But do we need to?

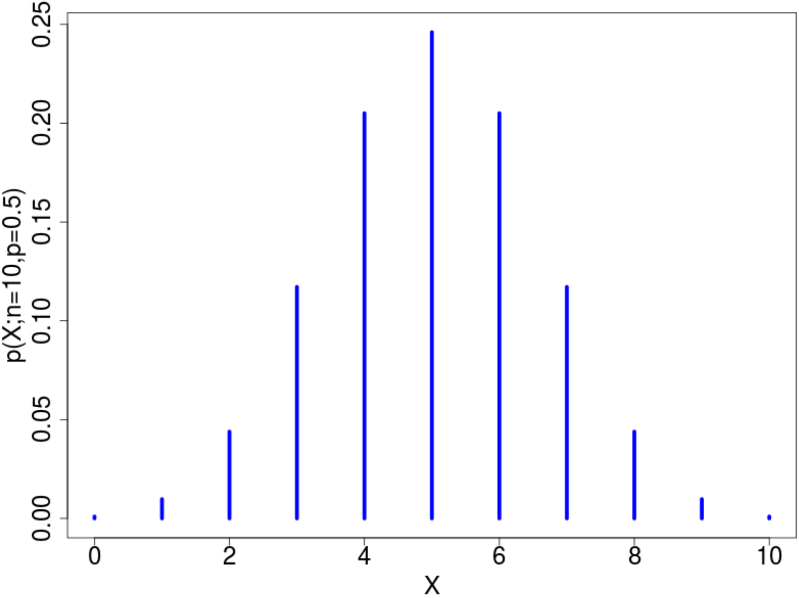
Normal approximation to the Binomial

- If $X \sim Bin(n, p)$ then X is approximately distributed as $N(\mu, \sigma^2)$ when n is large
- With $\mu = np$ and $\sigma^2 = np(1-p)$
- What is large? Rule of thumb: $\mu \geq 3\sigma$
- What is meant by approximately? $P(a < X < b)$ is close to $P(a < \mu + \sigma Z < b)$ for $Z \sim N(0, 1)$

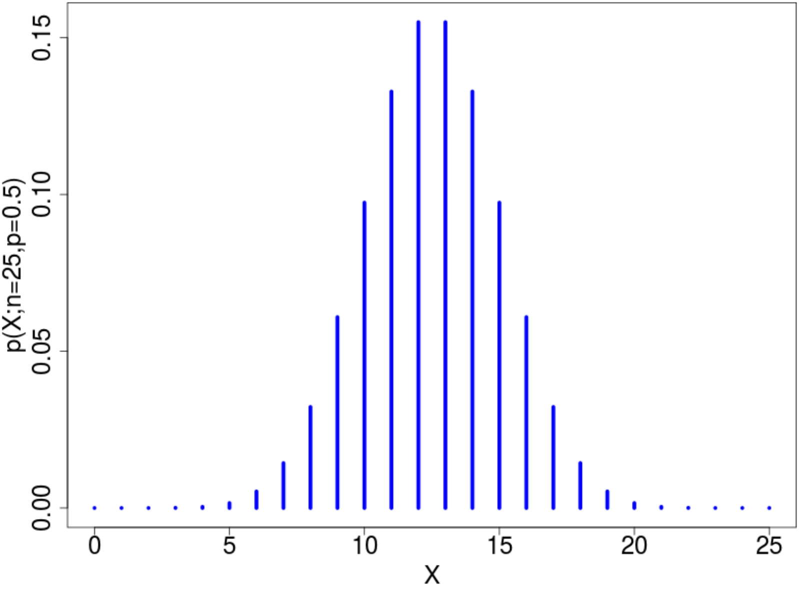
Binom(n=3,p=0.5)



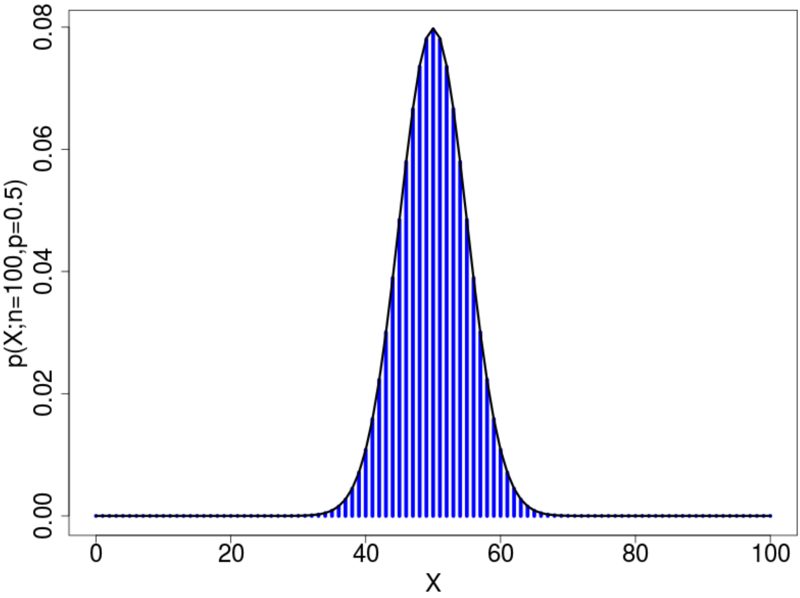
Binom(n=10,p=0.5)



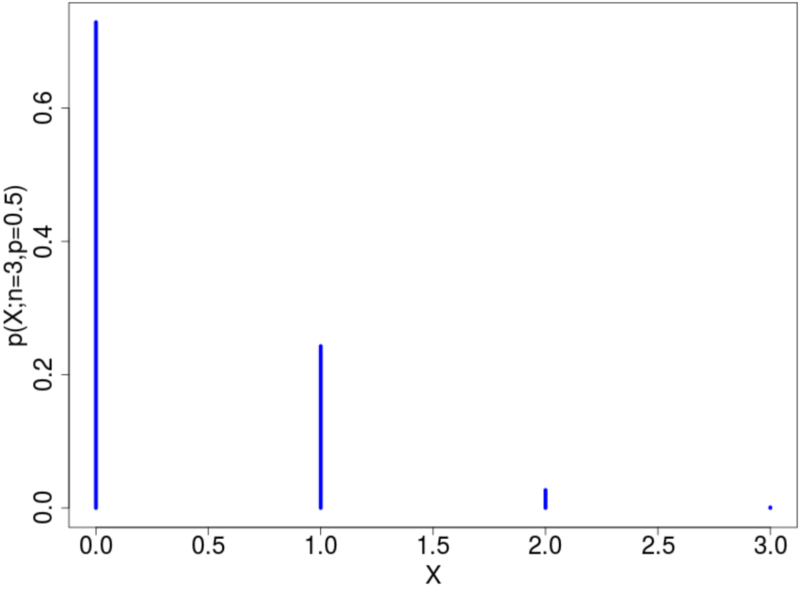
Binom(n=25,p=0.5)



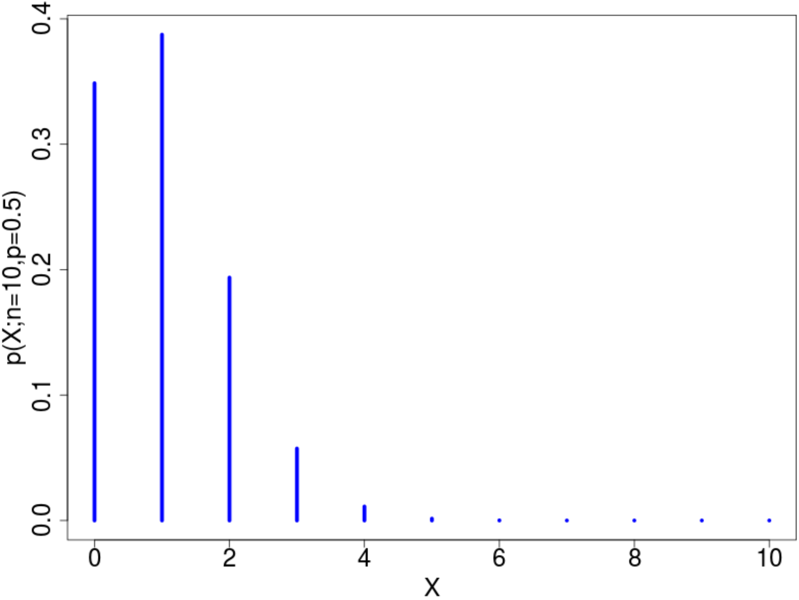
Binom(n=100,p=0.5)



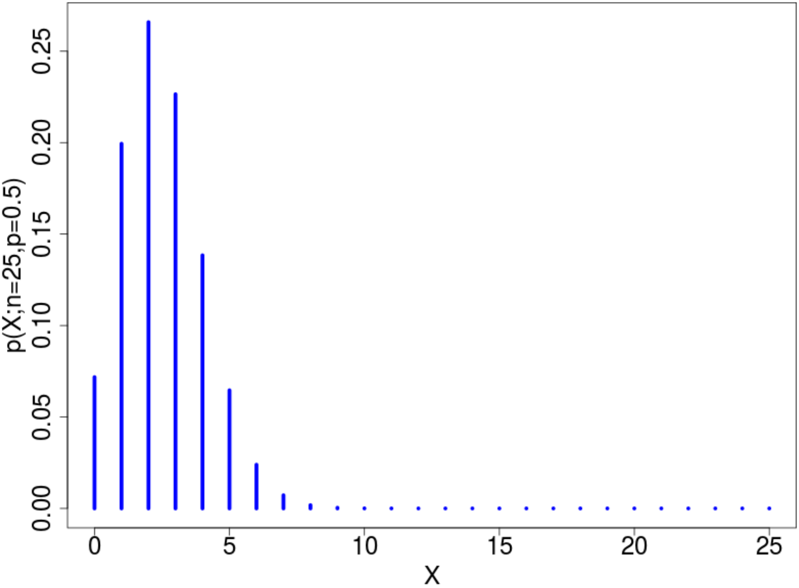
Binom(n=3,p=0.1)



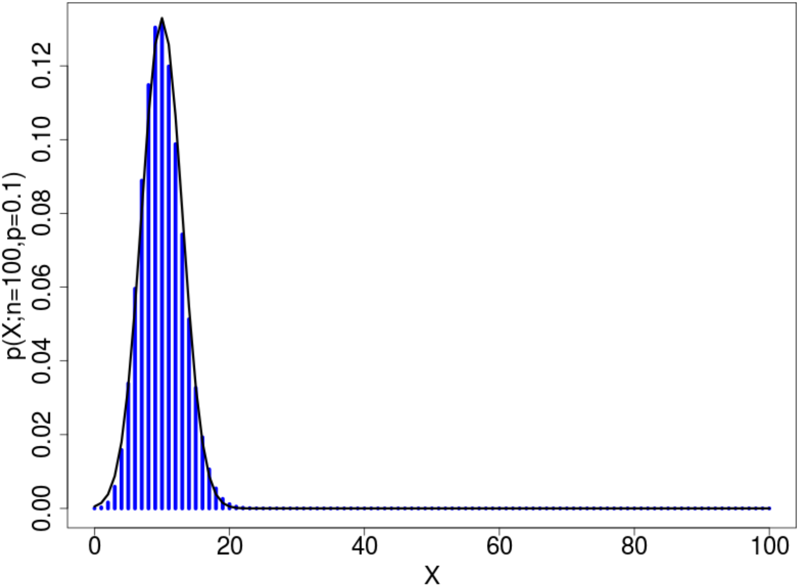
Binom(n=10,p=0.1)



Binom(n=25,p=0.1)



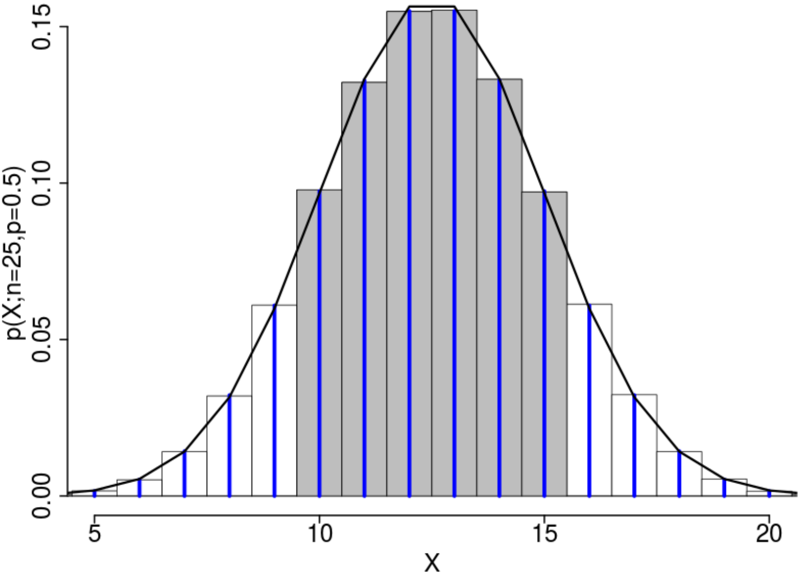
Binom(n=100,p=0.1)



Continuity Correction

- Suppose we wanted to know $p(10 \leq X \leq 15; n = 25, p = 0.5)$
- e.g. Probability that we get between 10 and 15 heads from 25 flips of a fair coin
- How do we account for 'discreteness' of X when using the normal approximation?

Continuity Correction



Continuity Correction

- Suppose we wanted to know $p(10 \leq X \leq 15; n = 25, p = 0.5)$
- e.g. Probability that we get between 10 and 15 heads from 25 flips of a fair coin
- How do we account for 'discreteness' of X when using the normal approximation?
- Answer: 'Continuity correct' to halfway between discrete values
- $P(a \leq X \leq b) \approx P(a - 0.5 \leq Y \leq b + 0.5)$, where $Y \sim N(\mu, \sigma^2)$

Continuity Correction

- Suppose we wanted to know $p(10 \leq X \leq 15; n = 25, p = 0.5)$
- e.g. Probability that we get between 10 and 15 heads from 25 flips of a fair coin
- How do we account for 'discreteness' of X when using the normal approximation?
- Answer: 'Continuity correct' to halfway between discrete values
- $P(a \leq X \leq b) \approx P(a - 0.5 \leq Y \leq b + 0.5)$, where $Y \sim N(\mu, \sigma^2)$
- Exact calulation: 0.7705

Continuity Correction

- Suppose we wanted to know $p(10 \leq X \leq 15; n = 25, p = 0.5)$
- e.g. Probability that we get between 10 and 15 heads from 25 flips of a fair coin
- How do we account for 'discreteness' of X when using the normal approximation?
- Answer: 'Continuity correct' to halfway between discrete values
- $P(a \leq X \leq b) \approx P(a - 0.5 \leq Y \leq b + 0.5)$, where $Y \sim N(\mu, \sigma^2)$
- Exact calulation: 0.7705
- Normal approximation: 0.7699

Continuity Correction

- Suppose we wanted to know $p(10 \leq X \leq 15; n = 25, p = 0.5)$
- e.g. Probability that we get between 10 and 15 heads from 25 flips of a fair coin
- How do we account for 'discreteness' of X when using the normal approximation?
- Answer: 'Continuity correct' to halfway between discrete values
- $P(a \leq X \leq b) \approx P(a - 0.5 \leq Y \leq b + 0.5)$, where $Y \sim N(\mu, \sigma^2)$
- Exact calulation: 0.7705
- Normal approximation: 0.7699
- Relative error: $=$ (exact-normal)/exact $= 0.0008$

Z test for proportions

- n independent trials with success probability p
- Observe X successes
- H0: Probability of success is p_0
- IF H0 is true, $X/n \sim N(p_0, \sqrt{p_0(1-p_0)/n})$
- Z test with $Z = \frac{X/n - p_0}{\sqrt{p_0(1-p_0)/n}}$

Z test for proportions

- n independent trials with success probability p
- Observe X successes
- H0: Probability of success is p_0
- IF H0 is true, $X/n \sim N(p_0, \sqrt{p_0(1-p_0)/n})$
- Z test with $Z = \frac{X/n - p_0}{\sqrt{p_0(1-p_0)/n}}$
- (Should do continuity correction, but this is not important for large n...)

Example: ESP

- Charles Tart (1970s): 7500 (500x15) attempts on 'Aquarius machine'
- Subjects predict which of four lights will come on
- Signal tells them if they were right.
- 7500 attempts. Expect 7500/4=1875 right.
- Actually observed 2006 correct guesses. Could it be purely by chance?

Example: ESP

- H0: Probability of success $p = 0.25$
- H1: Probability of success $p > 0.25$
- Significance level 0.01, meaning $Z_{crit} = 2.3$
- $X = 2006/7500 = 0.26747$, $\sigma = \sqrt{0.25 \times 0.75/7500} = 0.005$

$$Z = \frac{0.26747 - 0.25}{0.005} = 3.49$$

- Comfortably above critical value. $p = 0.0002$

ESP conclusions

Did the subjects do so well purely by chance?

- Almost certainly not. Under the null this would happen in one experiment out of 5000.
- Should we conclude that some of the subjects had the power to see into the future and predict which light would come on?
- Can you think of other alternatives?

ESP conclusions

Did the subjects do so well purely by chance?

- Almost certainly not. Under the null this would happen in one experiment out of 5000.
- Should we conclude that some of the subjects had the power to see into the future and predict which light would come on?
- Can you think of other alternatives?
- In fact, there was a problem with the machine which made the order of the lights be not independent.

Conclusion: You have to be careful in interpreting the results of statistical tests. Just because you can show it didn't happen 'by chance' doesnt mean your favourite alternative holds.

Subsection 3

The Z test for the difference between means

Z test for difference between means

- Observations from two different populations.
- Means from both are normally distributed.
- SDs are known: σ_1 and σ_2
- Unknown means μ_1 and μ_2
- Observe mean $\overline{X}_1$ from n_1 pop 1 samples and mean $\overline{X}_2$ from n_2 pop 2 samples
- Test H0 : $\mu_1 = \mu_2$
- If H0 is true, $X_1 - X_2 \sim N(0, \sigma^2)$
- with $\sigma = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$
- Test statistic $Z = (\overline{X}_1 - \overline{X}_2)/\sigma$

Example: Do tall men get picked first?

Heights of K Husbands, by age of marriage

	Age of marriage	
	early (< 30)	late (≥ 30)
number	160	35
mean height (mm)	1735	1716

- Suppose we know that the standard deviation of height is 70mm
- $\sigma = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} = 13.1$

$$Z = \frac{1735 - 1716}{13.1} = 1.45$$

- One tailed p-value 0.0735

Example: Do tall men get picked first?

Heights of K Husbands, by age of marriage

	Age of marriage	
	early (< 30)	late (≥ 30)
number	160	35
mean height (mm)	1735	1716

- Suppose we know that the standard deviation of height is 70mm
- $\sigma = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} = 13.1$

$$Z = \frac{1735 - 1716}{13.1} = 1.45$$

- One tailed p-value 0.0735
- Conclusion: Insufficient evidence to reject the null

Example: Do tall men get picked first?

Heights of K Husbands, by age of marriage

	Age of marriage	
	early (< 30)	late (≥ 30)
number	160	35
mean height (mm)	1735	1716

- Suppose we know that the standard deviation of height is 70mm
- $\sigma = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2} = 13.1$

$$Z = \frac{1735 - 1716}{13.1} = 1.45$$

- One tailed p-value 0.0735
- Conclusion: Insufficient evidence to reject the null
- Weitzman & Conley: "From Assortative to Ashortative Coupling: Men's Height, Height Heterogamy, and Relationship Dynamics in the United States": Short men tend to get married later ... but to stay married longer...

Subsection 4

Z test for the difference between proportions

Z test for difference between proportions

- Observations from two different kinds of trials.
- Probabilities of success are p_1 and p_2
- Test H0 : $p_1 = p_2$
- Observe X_1 successes from n_1 trials from pop 1
- Observe X_2 successes from n_2 trials from pop 2
- Standardized test statistic:

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

- with $\hat{p}_1 = X_1/n_1$, $\hat{p}_2 = X_2/n_2$ and $\hat{p} = (X_1 + X_2)/(n_1 + n_2)$

Example: Circumcision and AIDS

Study in Uganda: 70 circumcised men, 54 controls.

	circum.	non-circum.
n	70	54
infected	11	4

- $\hat{p}_1 = 11/70 = 0.157$
- $\hat{p}_2 = 4/54 = 0.121$
- $\sigma = \sqrt{0.121 \times 0.157 \left(\frac{1}{70} + \frac{1}{54}\right)} = 0.059$
- $Z = \frac{0.157 - 0.121}{0.059} = 1.41$
- One tailed p-value 0.08

Example: Circumcision and AIDS

Study in Uganda: 70 circumcised men, 54 controls.

	circum.	non-circum.
n	70	54
infected	11	4

- $\hat{p}_1 = 11/70 = 0.157$
- $\hat{p}_2 = 4/54 = 0.121$
- $\sigma = \sqrt{0.121 \times 0.157 \left(\frac{1}{70} + \frac{1}{54}\right)} = 0.059$
- $Z = \frac{0.157 - 0.121}{0.059} = 1.41$
- One tailed p-value 0.08
- Conclusion: Insufficient evidence to reject the null

Section 3

The t test

Subsection 1

The simple t test

Example: Kidney Dialysis

- Phosphate measure in blood of dialysis patients on six successive visits. Known to vary approximately according to normal distribution.
- One patient had the following measures in mg/dl:

$$5.6, 5.1, 4.6, 4.8, 5.7, 6.4$$

- Suppose 4.0 or below is a dangerous level.
- Test at the 0.01 level whether the level might be that low.
- $\overline{X} = 5.4$mg/dl (empirical mean)
- $s = 0.67$mg/dl (empirical standard deviation)

Example: Kidney Dialysis

- Problem! We don't know the SD!
- Estimate from the data:

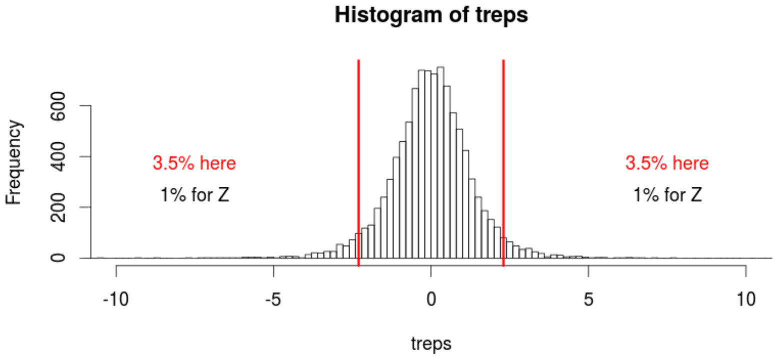
$$SD = \sigma \approx s = \sqrt{\frac{1}{n-1}\sum(X_i - \overline{X})^2}$$

- $Z = \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}}$
- $T = \frac{\overline{X}-\mu_0}{s/\sqrt{n}}$
- T has a different distribution to Z
- For $s = \sigma$ they are the same...
- But sample variation in s leads to T having 'heavier tails'
- $n - 1$ = number of degrees of freedom

Does it matter? Monte Carlo experiment

Critical value for Z at 0.01 level is 2.3.

1. Compute 6 independent samples from $N(4.0, 0.67^2)$
2. With mean $\overline{X}_j$ and empirical variance s_j
3. Compute $T_j = (\overline{X}_j - 4.0)/(s_j/\sqrt{6})$
4. Repeat 10000 times
5. Did T exceed 2.3 about 1% of the time?



The t test

- Suppose 4.0 or below is a dangerous level.
- Want a 1% chance of failing to recognise that the level is low
- H0: Average phosphate level $= 4.0$ mg/dl
- H1: Average phosphate level > 4.0 mg/dl

$$T = \frac{\overline{X} - \mu}{s/\sqrt{n}} = \frac{5.4 - 4.0}{0.67/\sqrt{6}} = 5.12$$

- Matlab: tinv(0.99,5)=3.36
- i.e. the critical value is 3.36
- Observed T is bigger - reject H0

The simple t test (Student's t test)

For n data points X_i

- If the mean of the data can be treated as Normal ...
- And our null hypothesis is $\overline{X} = \mu$ for known μ...
- When we estimate the standard deviation σ using s...
- We compute the standard error $SE = s/\sqrt{n}$
- We compute the test statistic $T = (\overline{X} - \mu)/SE$...
- Under the null, $T \sim t(0, 1, df = n - 1)$

So the t test (also) computes $p(T' \geq T)$ (one tailed)
Or $p(|T'| \geq |T|)$ (two tailed)
Important note: more generally, $df = n - p$ with p unknown parameters

Z test example

Time (min)	Sex	Weight (g)
5	F	3837
64	F	3334
78	M	3554
115	M	3838
177	M	3625
245	F	2208
247	F	1745
262	M	2846
271	M	3166
428	M	3520
455	M	3380
492	M	3294
494	F	2576
549	F	3208
635	M	3521
649	F	3746
653	F	3523
693	M	2902
729	M	2635
776	M	3920
785	M	3690
846	F	3430

Time (min)	Sex	Weight (g)
847	F	3480
873	F	3116
886	F	3428
914	M	3783
991	M	3345
1017	M	3034
1062	F	2184
1087	M	3300
1105	F	2383
1134	M	3428
1149	M	4162
1187	M	3630
1189	M	3406
1191	M	3402
1210	F	3500
1237	M	3736
1251	M	3370
1264	M	2121
1283	M	3150
1337	F	3866
1407	F	3542
1435	F	3278

- British birthweights have Mean 3426g and SD 538g from a large sample
- Australian birthweights sampled (1 day)
- Australians have mean 3276
- Null: birthweights have the same mean
- Alternative: Australian babies are smaller

t test example

Now imagine that we were not provided with the standard deviation of the British sample.

- Null hypothesis: The Australian birthweights are samples from the same distribution as UK birth weights.
- With one unknown parameter: σ
- If the null hypothesis holds,

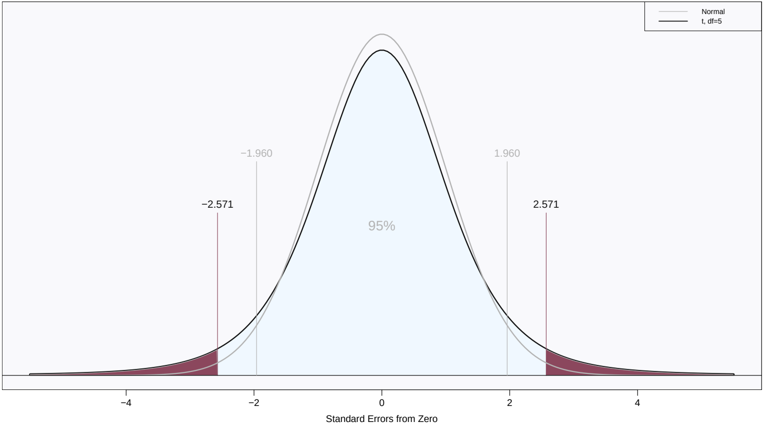
$$\overline{X} = (X_1 + \cdots + X_{44})/44 \sim N(3426, SE^2)$$

  with $SE = SD(X_i)/\sqrt{44} = 84.5$
- What is the probability that we would observe $\overline{X} \leq 3276$?
- Standardise: Observed $t = (\overline{x} - 3426)/84.5 \sim t(df = 44 - 1) = -1.64$

$p(t \leq -1.64) = 0.054$. Bigger than before:

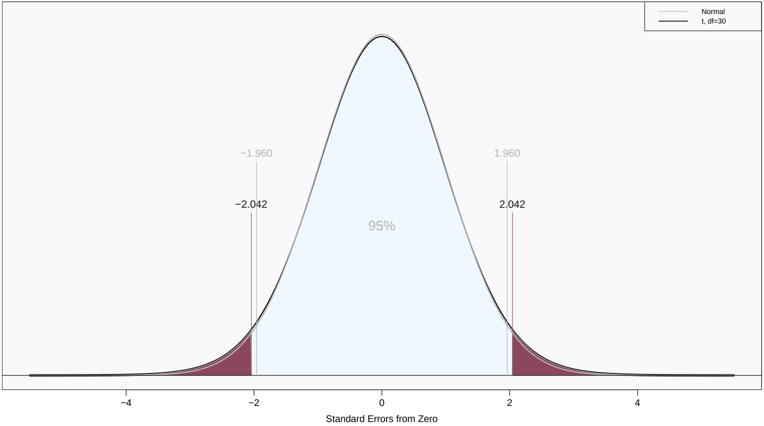
- Tails of $t(n)$ larger than tails of $N(0, 1)$
- Because of uncertainty in $\hat{\sigma}$

Tails of the student t-distribution



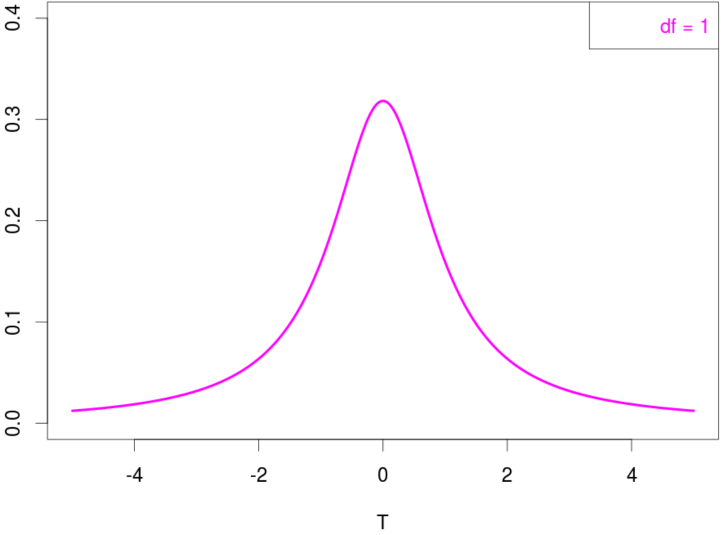
5 degrees of freedom

Tails of the student t-distribution

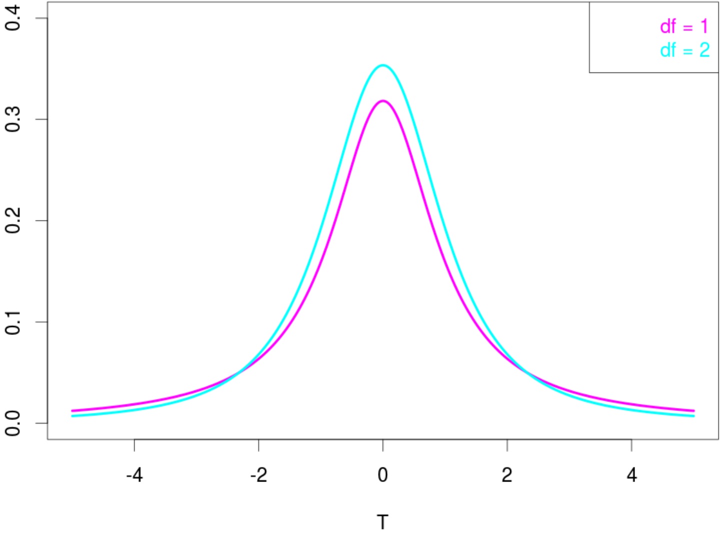


30 degrees of freedom

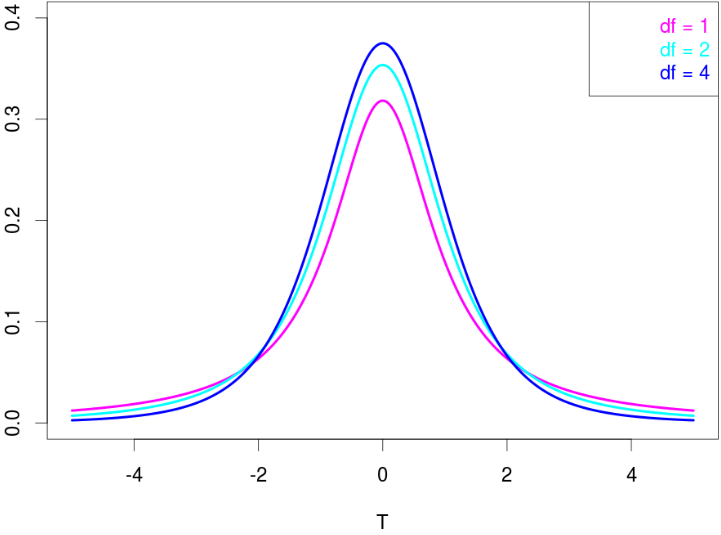
Students t distribution



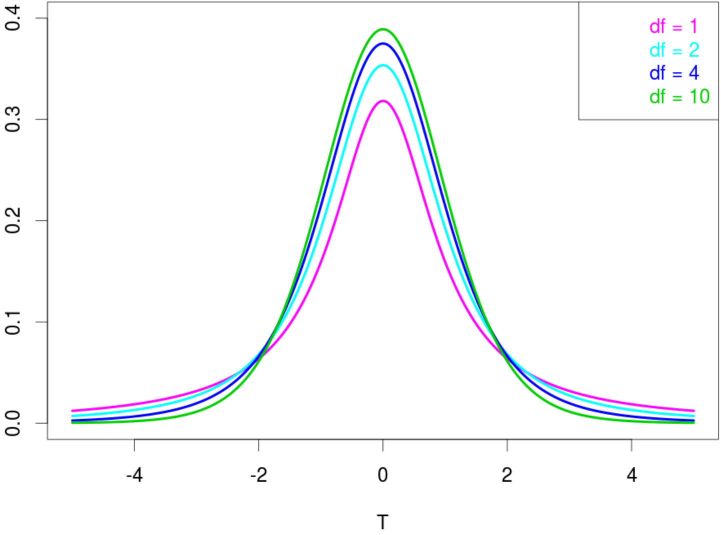
Students t distribution



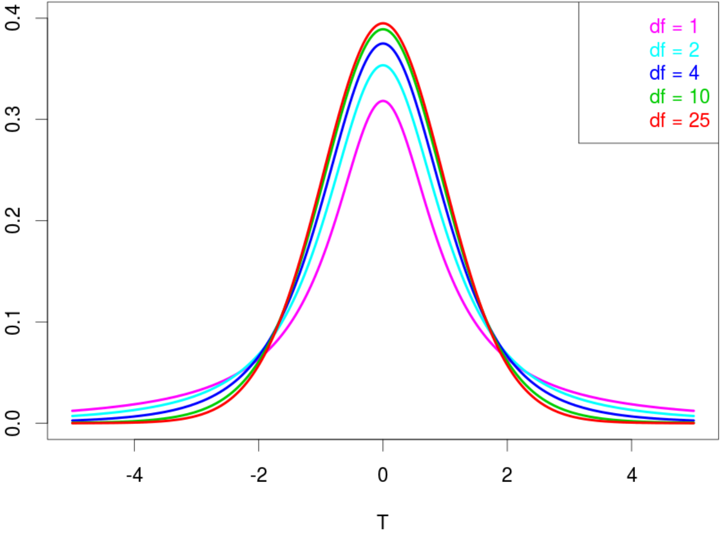
Students t distribution



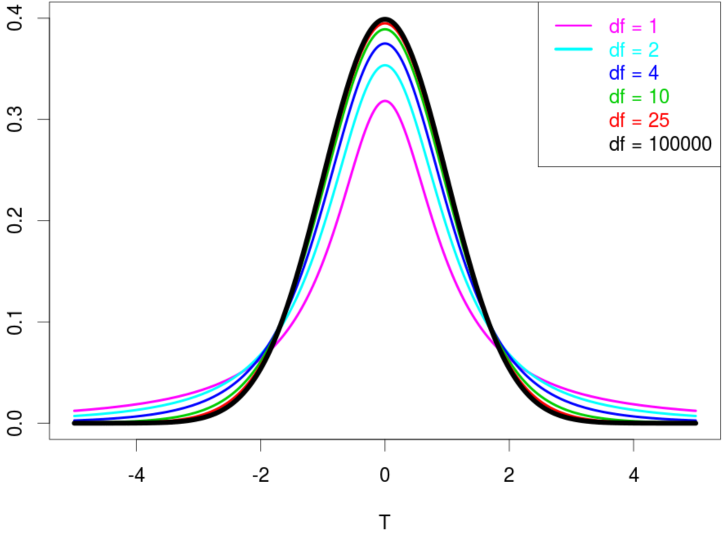
Students t distribution



Students t distribution



Students t distribution



Quiz time!

Let $t_\alpha(d)$ be the α quantile of the Student T distribution with d d.f. i.e. the probability of $t < \alpha$.

- $t_{0.95}(2) > t_{0.95}(3)$?
- $t_\alpha(100)$ is a little smaller than Z_α?
- 1% of the measurements of average phosphate will be within 2.3 s.d. of 4.0?

Quiz time!

Let $t_\alpha(d)$ be the α quantile of the Student T distribution with d d.f. i.e. the probability of $t < \alpha$.

- $t_{0.95}(2) > t_{0.95}(3)$?
- true
- $t_\alpha(100)$ is a little smaller than Z_α?

- 1% of the measurements of average phosphate will be within 2.3 s.d. of 4.0?

Quiz time!

Let $t_\alpha(d)$ be the α quantile of the Student T distribution with d d.f. i.e. the probability of $t < \alpha$.

- $t_{0.95}(2) > t_{0.95}(3)$?
- true
- $t_\alpha(100)$ is a little smaller than Z_α?
- false - it can be very different for small α
- 1% of the measurements of average phosphate will be within 2.3 s.d. of 4.0?

Quiz time!

Let $t_\alpha(d)$ be the α quantile of the Student T distribution with d d.f. i.e. the probability of $t < \alpha$.

- $t_{0.95}(2) > t_{0.95}(3)$?
- true
- $t_\alpha(100)$ is a little smaller than Z_α?
- false - it can be very different for small α
- 1% of the measurements of average phosphate will be within 2.3 s.d. of 4.0?
- false - that is only true for Z

Subsection 2

The Matched sample t test

Example - Schizophrenia

- 15 healthy, 15 Schizophrenia sufferers
- Measure hippocampus volume
- schizophrenic: 1.27,1.63,1.47,1.39,1.93,1.26, 1.71,1.67,1.28,1.85,1.02,1.34, 2.02,1.59,1.97
- healthy: 1.94,1.44,1.56,1.58,2.06,1.66, 1.75,1.77,1.78,1.92,1.25,1.93, 2.04,1.62,2.08
- Test for equality of means at 0.05 level.
- Dont know SD.

	Unaff.	Schiz.
Mean	1.76	1.56
SD	0.24	0.30

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

- Here $s_p = 0.27$

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

- Here $s_p = 0.27$
- Standard error $SE = s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}} = 0.099$

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

- Here $s_p = 0.27$
- Standard error $SE = s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}} = 0.099$
- $T = \frac{\overline{X} - \overline{Y}}{SE} = 2.02$

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

- Here $s_p = 0.27$
- Standard error $SE = s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}} = 0.099$
- $T = \frac{\overline{X} - \overline{Y}}{SE} = 2.02$
- $df = n_x + n_y - 2 = 28$

2-sample t test

- X: Schizophrenic data
- Y: Non-schizophrenic data
- H0 : Samples came from the same distribution.
- If H0 true, then we can estimate σ by pooling X and Y
- Pooled sample variance:

$$s_p^2 = \frac{(n_x - 1)s_X^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}$$

- Here $s_p = 0.27$
- Standard error $SE = s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}} = 0.099$
- $T = \frac{\overline{X} - \overline{Y}}{SE} = 2.02$
- $df = n_x + n_y - 2 = 28$
- Critical value at $p = 0.05$ is $T = 2.05$. Don't reject.

What does this mean?

- The schizophrenic subjects have smaller hippocampal volume on average.
- BUT there's a lot of variability overall - samples of 15 individuals can differ by this much purely by chance.
- Can we do anything to reduce this variability within groups, so we can see the difference between the groups more clearly?

Subsection 3

Matched sample t test

Matched case-control study

- Idea: Experiment and control group are in matched pairs, chosen to be similar in ways likely to affect what we're measuring.
- Why?
- A lot of the variability will disappear (we hope) from the difference, since the matched pairs will vary together.
- Shared variance cancels out!

Example: Schizophrenia

The 30 subjects in the schizophrenia study were 15 matched pairs of monozygotic twins.

- Mean difference $E(X - Y) = 0.20$ (as before, i.e. $E(X) - E(Y)$)
- However, standard deviation $s_{diff} = s(D) = s(X - Y) = 0.238$
- Now a standard t test:
- Test H0: $\mu_{diff} = 0$
- $T = \frac{E(X-Y)}{s_{diff}/\sqrt{15}} = 3.25$
- Critical T for $df = 14$ is 2.15
- Reject H0.

Section 4

The χ^2 test

Subsection 1

The χ^2 test

Example: suicides by birth month

Salib and Cortina-Borja examined death certificates of 26,886 suicides in England and Wales. Tabulated by month of birth.

Does spring birthday predispose to suicide?

Month	Female	Male	Total
Jan	527	1774	2301
Feb	435	1639	2074
Mar	454	1939	2393
Apr	493	1777	2270
May	535	1969	2504
Jun	515	1739	2254
Jul	490	1872	2362
Aug	489	1833	2322
Sep	476	1624	2100
Oct	474	1661	2135
Nov	442	1568	2010
Dec	471	1690	2161

Example: suicides by birth month

- Null hypothesis: Suicides are equally likely to have been born any day of the year. The probability of having been born in a given month is proportional to the number of days in the month.
- Test the null hypothesis at the 0.01 level.
- One approach: Divide into two groups, and use the Z test

Example: suicides by birth month

- Null hypothesis: Suicides are equally likely to have been born any day of the year. The probability of having been born in a given month is proportional to the number of days in the month.
- Test the null hypothesis at the 0.01 level.
- One approach: Divide into two groups, and use the Z test
- If we define spring as March–June, there are 122 days
- Under H0 P(spring birthday) $= 122/365.25 = 0.34$
- Observed number spring $= 9421$
- Expected number spring $= 0.334 \times 26886 = 8980$
- Standard Error $= \sqrt{0.334 \times 0.666 \times 26886} = 77.3$
- $Z = \frac{9421-8980}{77.3} = 5.71$
- Critical value $Z_{crit} = 2.58$
- p-value $\approx 10^{-8}$

Example: suicides by birth month

- Problem: We cheated!
- We used the largest choice of months
- Was that our hypothesis before we saw the data?
- No. We might have instead wondered if there were any months that were unusual
- Can we test all deviations simultaneously?

The χ^2 test

- Test statistic that measures deviations in all k categories simultaneously.
- $\chi^2 = \sum \frac{(\text{observed}_i - \text{predicted})^2}{\text{expected}}$
- $Z_i = \frac{\text{observed}_i - \text{predicted}}{\text{standard error}}$
- $\chi^2(k) = \sum_{k-1} Z_i^2$ where Z_i are independent
- Mathematical fact: If the number of observations is large then this chi-squared statistic has a certain distribution, called the χ^2 distribution.
- How large is large? Rule of thumb: At least 5 expected in each cell of the table.

The χ^2 test

- Test statistic that measures deviations in all k categories simultaneously.
- $\chi^2 = \sum \frac{(\text{observed}_i - \text{predicted})^2}{\text{expected}}$
- $Z_i = \frac{\text{observed}_i - \text{predicted}}{\text{standard error}}$
- $\chi^2(k) = \sum_{k-1} Z_i^2$ where Z_i are independent
- Mathematical fact: If the number of observations is large then this chi-squared statistic has a certain distribution, called the χ^2 distribution.
- How large is large? Rule of thumb: At least 5 expected in each cell of the table.
- This distribution also has a 'degrees of freedom' parameter
- General rule: df = Num. data pts - parameters estimated - 1

The χ^2 test

- Test statistic that measures deviations in all k categories simultaneously.
- $\chi^2 = \sum \frac{(\text{observed}_i - \text{predicted})^2}{\text{expected}}$
- $Z_i = \frac{\text{observed}_i - \text{predicted}}{\text{standard error}}$
- $\chi^2(k) = \sum_{k-1} Z_i^2$ where Z_i are independent
- Mathematical fact: If the number of observations is large then this chi-squared statistic has a certain distribution, called the χ^2 distribution.
- How large is large? Rule of thumb: At least 5 expected in each cell of the table.
- This distribution also has a 'degrees of freedom' parameter
- General rule: df = Num. data pts - parameters estimated - 1
- So here: df = cells - parameters estimated - 1

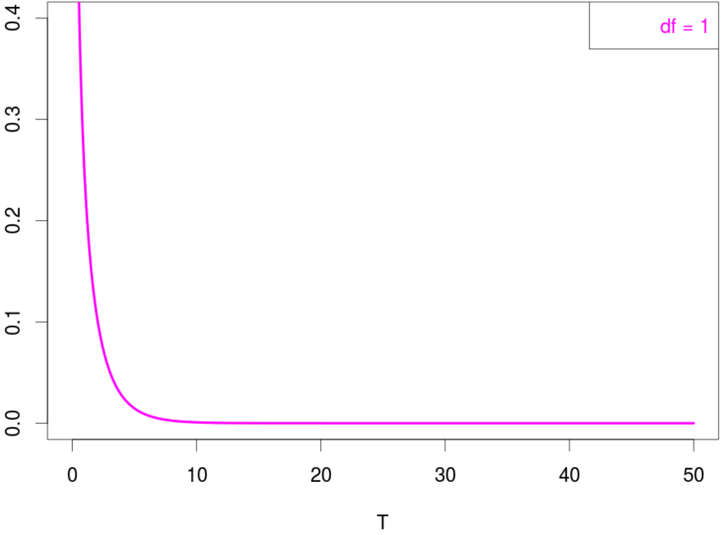
Calculating with χ^2

- χ^2 with d degrees of freedom has mean d and variance $2d$.
- It has a density proportional to

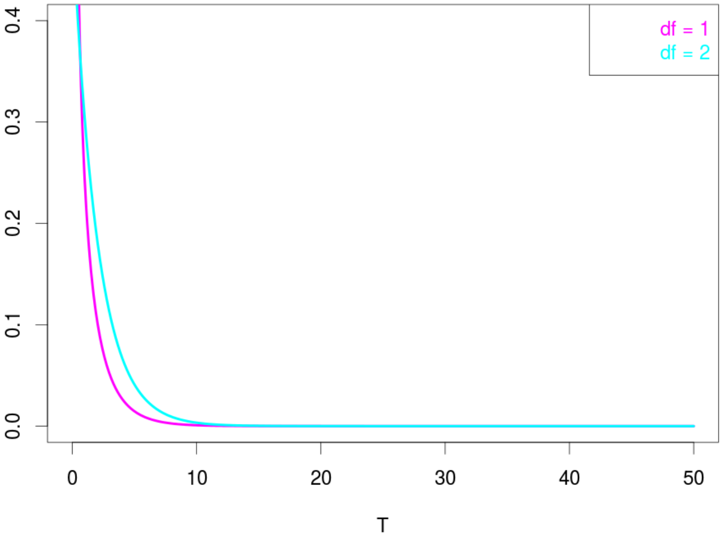
$$x^{d/2-1}e^{-x/2}$$

- We will not use this formula, instead using the computer (as usual!)

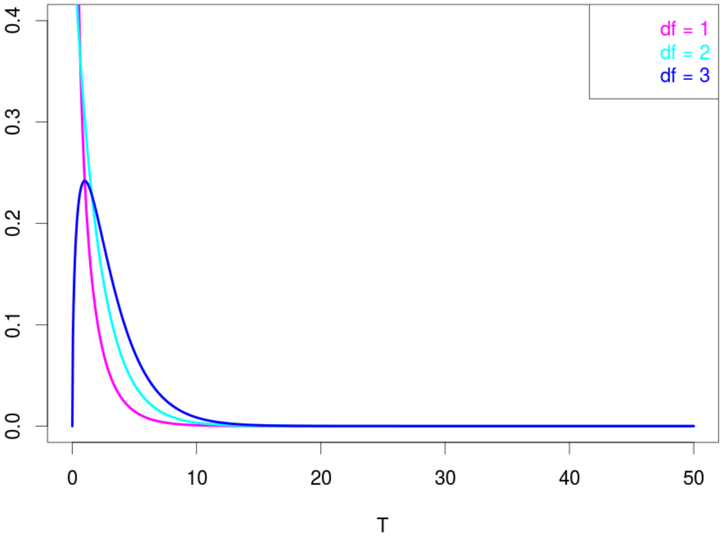
The χ^2 Distribution



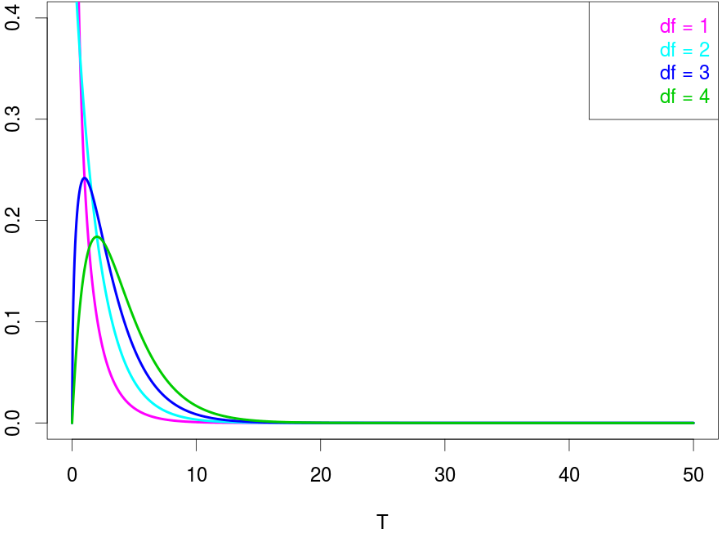
The χ^2 Distribution



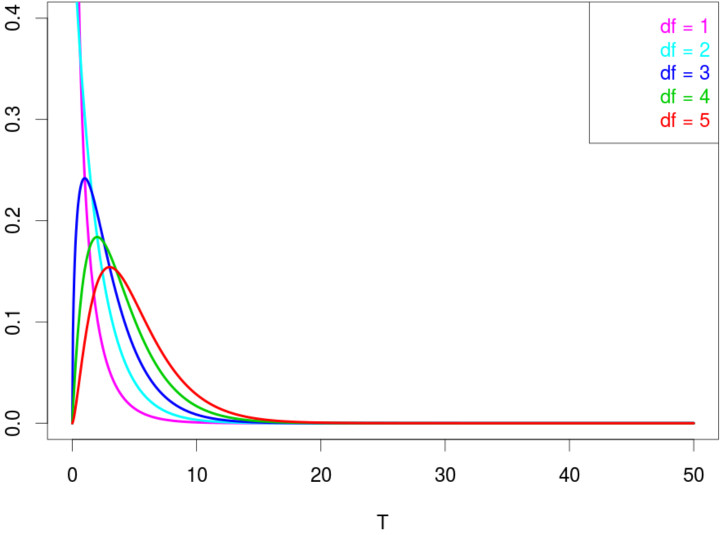
The χ^2 Distribution



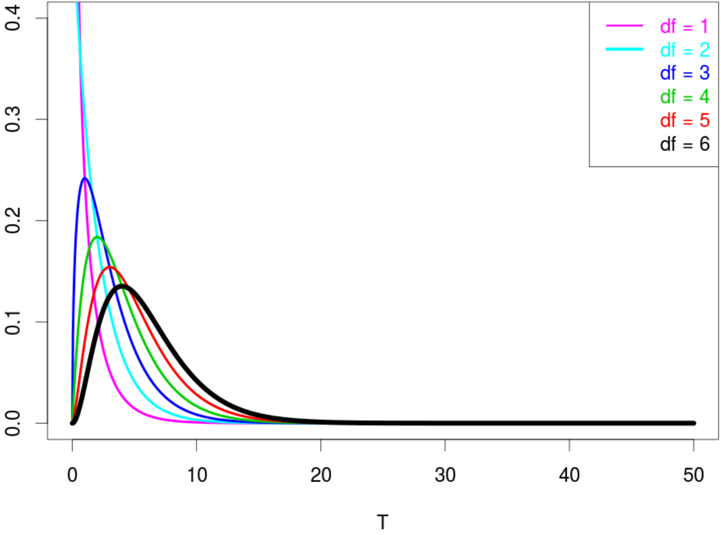
The χ^2 Distribution



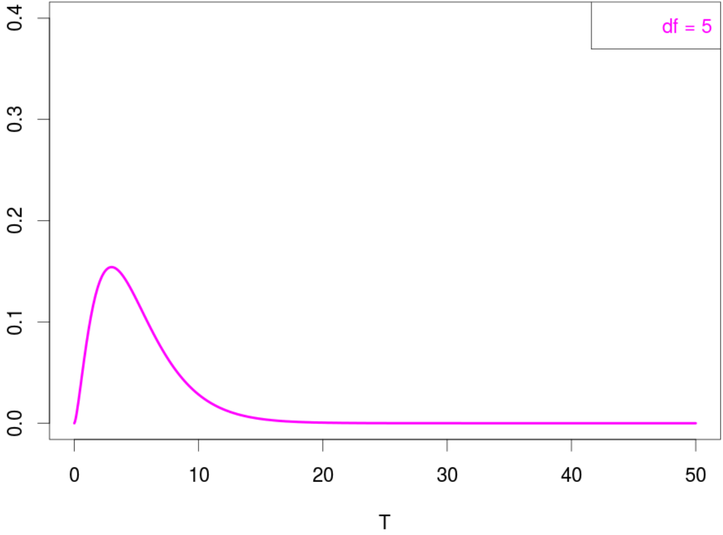
The χ^2 Distribution



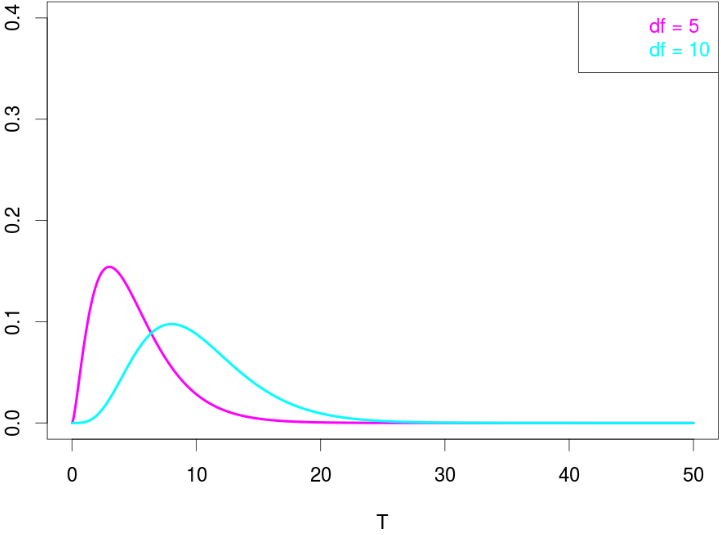
The χ^2 Distribution



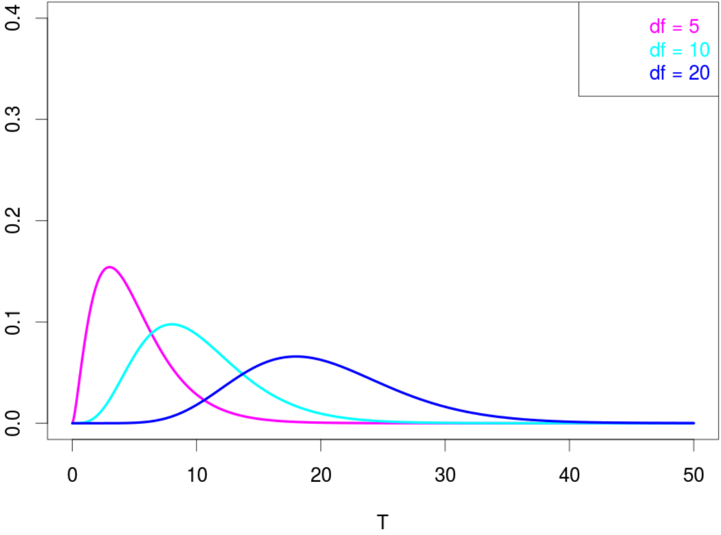
The χ^2 Distribution



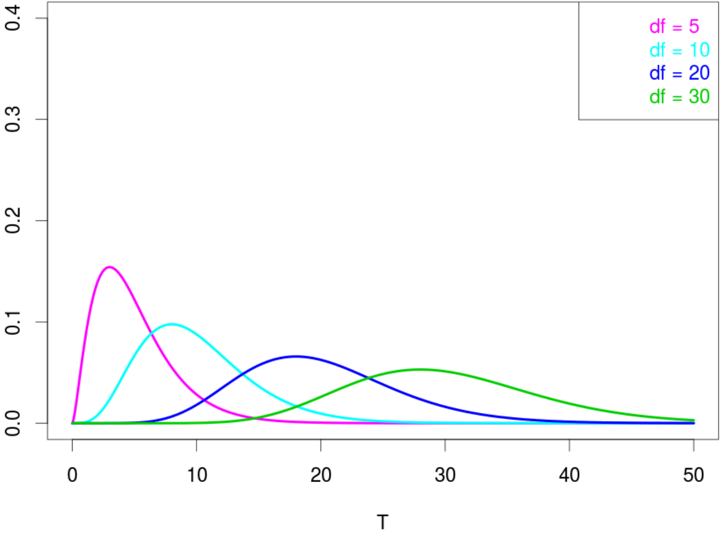
The χ^2 Distribution



The χ^2 Distribution



The χ^2 Distribution



Simple example: Testing a die

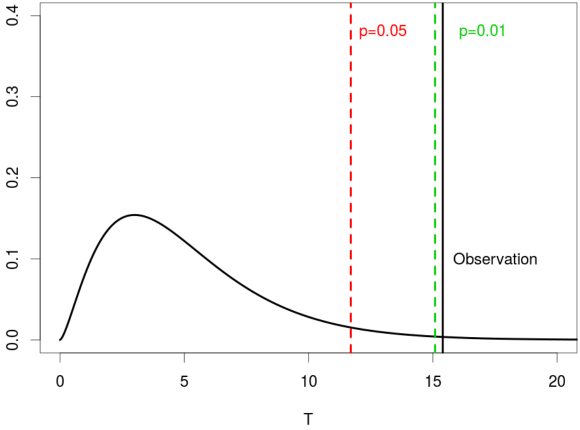
side	1	2	3	4	5	6
freq	16	15	4	6	14	5

- Test the null hypothesis that all sides are equally likely at the 0.01 level

$$
\begin{aligned}
X^2 &= \sum \frac{(\text{observed} - \text{expected})^2}{\text{expected}} && (1) \\
&= \frac{(16-10)^2}{10} + \cdots + \frac{(5-10)^2}{10} = 15.4 && (2)
\end{aligned}
$$

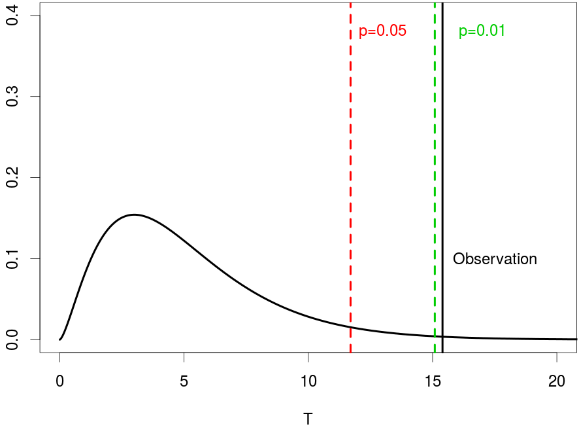
Simple example: Testing a die

- $X^2 = 15.4$ with 5 degrees of freedom
- Matlab: 1-chi2cdf(15.4,5)=0.00885



Simple example: Testing a die

- $X^2 = 15.4$ with 5 degrees of freedom
- Matlab: 1-chi2cdf(15.4,5)=0.00885
- Reject at the 0.01 level



Example: suicides by birth month

Salib and Cortina-Borja examined death certificates of 26,886 suicides in England and Wales. Tabulated by month of birth.

Does spring birthday predispose to suicide?

Month	Female	Male	Total
Jan	527	1774	2301
Feb	435	1639	2074
Mar	454	1939	2393
Apr	493	1777	2270
May	535	1969	2504
Jun	515	1739	2254
Jul	490	1872	2362
Aug	489	1833	2322
Sep	476	1624	2100
Oct	474	1661	2135
Nov	442	1568	2010
Dec	471	1690	2161

Suicides by birth month

- H0: Suicides are equally likely to have been born any day of the year. The probability of having been born in a given month is proportional to the number of days in the month.
- e.g., P(January)=31/365.25 = 0.0849
- P(February)=28.25/365.25 = 0.0773

$$
\begin{aligned}
X^2 &= \sum \frac{(\text{expected} - \text{observed})^2}{\text{expected}} && (3) \\
&= \frac{(2281 - 2301)^2}{2281} + \cdots + \frac{(2281 - 2161)^2}{2281} && (4) \\
&= 72.4 && (5)
\end{aligned}
$$

- df $= 12 - 1 = 11$
- p-value, from Matlab: 1-chi2cdf(72.4,11)=0.00000000004262901
- i.e. $p < 10^{-10}$. Reject H0.
- The variation is not due to chance variation.

Suicides by birth month

- Looking at just the data for females:

$$X^2 = \frac{(492-527)^2}{492} + \cdots + \frac{(492-471)^2}{492} \quad (6)$$
$$= 17.4 \quad (7)$$

- Matlab: chi2inv(.95,11)=19.68
- We do not reject H0 at the 5% level....
- "The difference in frequency of suicides by birth month among women is NOT statistically significant. It could be explained by chance variation."

Section 5

Non-parametric tests

Subsection 1

Why we need non-parametric tests

An experiment

- "If a newborn infant is held under his arms and his bare feet are permitted to touch a flat surface, he will perform well-coordinated walking movements similar to those of an adult... Normally, the walking and pacing reflexes disappear by about 8 weeks."
- Observation: If the infant exercises this reflex, it does not disappear.
- Hypothesis: Maintaining this reflex will help children learn to walk earlier.

How do we test this hypothesis?

- Idea: Do weekly exercises with a newborn. See when he/she starts walking.
- Result: 10 months.
- Problem: Is that long or short?
- New Idea: Do weekly exercises with a newborn. Don't do weekly exercises with another newborn. See which one starts walking first
- Result: mean with exercise 10.1 months
- without exercise 11.7 months
- Problem: Newborns don't all start walking at the same age, regardless of exercise.

t test for walking data

Age in months at first walking

Treatment (Exercise)	Control (No Exercise)
9.0	11.5
9.5	12.0
9.75	9.0
10.0	11.5
13.0	13.25
9.5	13.0
Mean 10.1	11.7
SD 1.45	SD 1.52

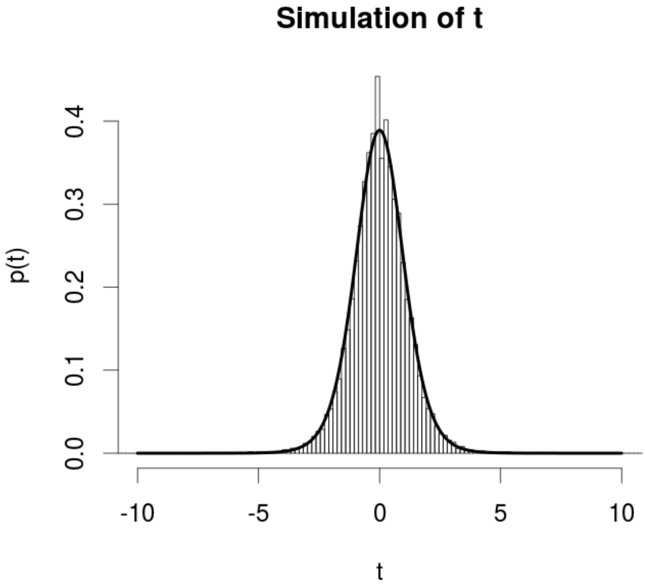
t test for walking data

- The Treatment numbers are generally smaller, but not always. Could the difference be merely due to chance?
- Two sample t test
- H0: $\mu_T = \mu_C$
- H1: $\mu_T < \mu_C$ (one-tailed test)
- Test at 0.05 signif. level 12-2=10 d.f.
- Critical value 1.81
- Pooled sample variance: $s_p = \sqrt{\frac{(6-1)1.45^2+(6-1)1.52^2}{6+6-2}} = 1.48$
- Standard error $SE = s_p\sqrt{1/6 + 1/6} = 0.85$
- $T = \frac{\overline{X}-\overline{Y}}{SE} = 1.85$
- Reject Null

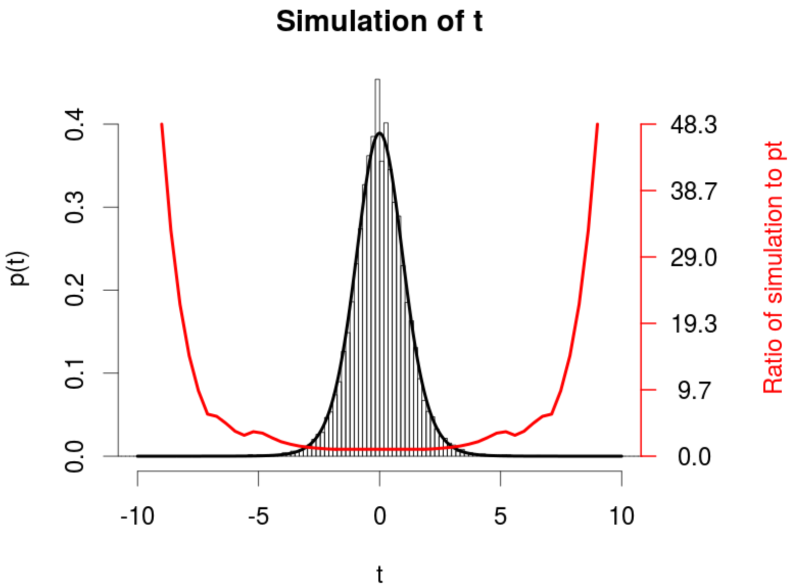
What if the distribution isn't Normal?

- Under the null...
- A bimodal distribution?
- Mean of 6 samples will not be Normal!

Simulation study using replicate data



Simulation study using replicate data



Nonparametric tests

- Idea: Come up with test statistics whose significance level doesn't depend on the distribution that the data came from.
- Advantage: We reject with the right probability if the null hypothesis is true.
- Drawback: We lose power. That is, we need a larger sample to reject the null if its false.
- We focus on two tests that the median of two distributions is equal
- They are varyingly sensitive to other differences in distribution

Subsection 2

Mann-Whitney U test

Mann-Whitney U test

Also called the Wilcoxon two sample Rank-sum test

- We have samples $X_1, \cdots, X_{n_x}$ and $Y_1, \cdots, Y_{n_y}$ with unknown distributions.
- H0: medians are the same.
- H1 :$m_x > m_y$ (one-tailed) or $m_x \neq m_y$ (two-tailed)

Mann-Whitney calculation

- Step 1: Put the data in order: $Z_i = \mathrm{sort}(X, Y)$
- Step 2: Write down the ranks: $r_i = i$
- Step 2.5: Combine ties: $r_i = \mathrm{mean}(r_j | Z_j = Z_i)$
- Step 3: Add up the ranks: $R_x = \sum_{i=1 \mathrm{s.t.} i \in X}^{N} r_i$, $R_Y = \sum_{i=1 \mathrm{s.t.} i \in Y}^{N} r_i$
- Step 4: Compute $R = \min(R_X, R_Y)$
- Step 5: Compute significance. Under H0, each ranking should be random from (1,N) with replacement.
- Matlab: p=ranksum(X,Y)
- This computes all permutations for small N, and uses the normal approximation to the sum for large N

Computation for walking babies

9.0	9.0	9.5	9.5	9.75	10.0	11.5	11.5	12	13.0	13.0	13.25
1.5	1.5	3.5	3.5	5	6	7.5	7.5	9	10.5	10.5	12

- $R_X = 30$, $R_Y = 48$
- $R = R_X = 30$ since X and Y have the same size
- p=ranksum(X,Y,'tail','left') = 0.085 (recent matlab versions only!)
- p=ranksum(X,Y)/2 = 0.085 (old versions only implement two-tailed test)
- Retain the null hypothesis

Subsection 3

Paired value tests

The sign test

- We have paired samples $X_1, Y_1, \cdots, X_n, Y_n$
- Work only with $I_i = \mathcal{I}(X_i - Y_i > 0)$, i.e. 1 if $X_i > Y_i$, 0 otherwise.
- H0: $S = \sum_i I_i$ is a Binomial RV with success probability $p = 0.5$
- H1: $p > 0.5$
- Matlab: $\mathrm{binocdf}(S, n, 0.5)$ (left tail)
- Matlab: $\mathrm{binocdf}(n - S, n, 0.5)$ (right tail)
- Matlab: $2\min(\mathrm{binocdf}(S, n, 0.5), \mathrm{binocdf}(n - S, n, 0.5))$ (two-tailed)

Example: Schizophrenia

s.	1.27	1.63	1.47	1.39	1.93	1.26	1.71	1.67	1.28	1.85	1.02	1.34	2.02	1.59	1.97
h.	1.94	1.44	1.56	1.58	2.06	1.66	1.75	1.77	1.78	1.92	1.25	1.93	2.04	1.62	2.08
d.	+	-	+	+	+	+	+	+	+	+	+	+	+	+	+

- Observed $X > Y$ for 14 out of 15 twins
- Compute $P(S = 14, 15) = \left({}^{15}C_{14} + {}^{15}C_{15}\right)\left(\frac{1}{2}\right)^{15} = 0.0005$
- Or we can use the Normal approximation, p-value $p = 0.0004$:

$$Z = \frac{14 - 0.5 \times 15}{\sqrt{0.5 \times 0.5 \times 15}} = 3.36$$

Wilcoxon one sample sign-rank test

- Idea: It makes sense to consider, not just if $X > Y$, but whether this happens for big or small numbers.
- It might be that there are equal numbers of $+$ and $-$ differences, but the $+$ are bigger
- Step 1: list differences ordered by absolute value
- Step 2: Calculate $W = \min(\sum_{i:X>Y}(r_i), \sum_{i:X<Y}(r_i))$
- Step 3: Compare W to the distribution that would be given under the null, that the ranks are unrelated to the signs
- For small n this means enumerating all options (enumeration is not fun)

Wilcoxon one sample sign-rank test

- For $n \approx 10$ or more, $Z = \frac{W - \mu_W}{\sigma_W} \sim N(0, 1)$, and a Z test can be used,
- Where

$$\mu_W = \frac{1}{2} + \frac{n(n+1)}{4}$$

and

$$\sigma_W = \sqrt{\frac{n(n+1)(2n+1)}{24}}.$$

Example: Schizophrenia

s.	1.27	1.63	1.47	1.39	1.93	1.26	1.71	1.67	1.28	1.85	1.02	1.34	2.02	1.59	1.97
h.	1.94	1.44	1.56	1.58	2.06	1.66	1.75	1.77	1.78	1.92	1.25	1.93	2.04	1.62	2.08
d.	0.67	-0.19	0.09	0.19	0.13	0.40	0.04	0.10	0.50	0.07	0.23	0.59	0.02	0.03	0.11

Leading to the sorted differences

0.02	0.03	0.04	0.07	0.09	0.10	0.11	0.13	-0.19	0.19	0.23	0.40	0.50	0.59	0.67
1	2	3	4	5	6	7	8	9.5	9.5	11	12	13	14	15

- Sums: $R_x = 110.5,\ R_Y = 9.5$
- So $R = 9.5$
- $\mu_R = 0.5 + 15 \times 16/4 = 60.5$
- $\sigma_R = \sqrt{15 \times 16 \times (30+1)/24} = 17.6$
- $Z = (R - \mu_R)/\sigma_R = -2.897$
- So the p-value is 0.0019

Section 6

The menagerie of tests

But which test should I use???

- These are just a small subset of the possible tests available
- Many of them have different options:
  - How did we make the data look like a Normal?
  - Parameters in model - leads to degrees of freedom
  - Tails of test
  - Pairing data
  - etc
- How to decide?

Model assumptions

The key is to make appropriate assumptions

- Are your data independent and random samples from a defined population? (All tests considered here)
- Are you primarily testing for a difference in the location in the two distributions? (Z,t,non-parametric tests)
- Or the variance of many random variables? (χ^2)
- Is the underlying distribution normal? (Z test, t test, χ^2 test)
- Or do we want to avoid assumptions about it, and test the median? (Mann-Whitney, Wilcoxon)
- If so, do we want to test the whole distribution? (Wilcoxon)
- Are there unknown parameters? (t test, χ^2 test)

Look up the specific assumptions when you use a test!

Z Test

H1 tests for a difference in the mean of two distributions.
H0 makes the following assumptions:

- Independent random samples
- Mean is approx. normal
- Continuous variable (recall: continuity correction)
- Known variance

t Test

H1 tests for a difference in the mean of two distributions.
H0 makes the following assumptions:

- Independent random samples
- Mean is approx. normal
- Continuous variable (recall: continuity correction)
- Unknown variance, estimated using s

χ^2 Test

H1 tests for a difference in the variance of n distributions.
H0 makes the following assumptions:

- Sum of independent random samples
- Whose mean is approx. normal (hence sample size > 5 desirable)
- Continuous variable (recall: continuity correction)
- Unknown variance, target of the test

Sign Test

H1 tests for a difference in the median of two distributions.
H0 makes the following assumptions:

- Independent random samples

Mann-Whitney U Test

H1 tests for a difference in the median of two distributions.
H0 makes the following assumptions:

- Independent random samples
- Continuous variable (recall: tied value correction)

This is 'just' the unpaired Wilcoxon Test.

Wilcoxon Test

H1 tests for a difference in either the median of two paired distributions.
H0 makes the following assumptions:

- Independent random samples
- Continuous variable (recall: tied value correction)
- symmetric distribution of differences

Other tests you might encounter

We have looked at tests for the location of one or more distributions. Other important cases are:

- F-test: Compares the variance of two distributions. Used in Analysis of variance (ANOVA).
- Kolmogorov-Smirnov test: A non-parametric test for whether two distributions are the same, based on the maximum deviation from the empirical cumulative density functions.

Conceptually different tests

Conceptually different tests

- Likelihood ratio test: the most important, because it uses a specific alternative hypothesis. It considers two models, one of which can be more complicated than the other (but nested). It accounts for the difference in complexity. But: you have to define the two models explicitly.

Conceptually different tests

- Likelihood ratio test: the most important, because it uses a specific alternative hypothesis. It considers two models, one of which can be more complicated than the other (but nested). It accounts for the difference in complexity. But: you have to define the two models explicitly.
- Monte carlo tests: If we don't know the distribution of the data, but can simulate from it, we can simulate $k-1$ test statistics and report the p-value as the quantile of the true test.

Conceptually different tests

- Likelihood ratio test: the most important, because it uses a specific alternative hypothesis. It considers two models, one of which can be more complicated than the other (but nested). It accounts for the difference in complexity. But: you have to define the two models explicitly.
- Monte carlo tests: If we don't know the distribution of the data, but can simulate from it, we can simulate $k-1$ test statistics and report the p-value as the quantile of the true test.
- Bayesian tests: A very different paradigm, Bayesian tests usually ask whether a parameter estimate falls outside of some range, given the data and some prior knowledge of the parameter.