This paper contains two sections, Section A and Section B. Answer each section in a separate answer book.

Section A contains TEN short questions worth 4 marks each. All TEN answers will be used for assessment.

Section B contains SIX longer questions worth 15 marks each. A candidate’s FOUR best answers will be used for assessment.

Calculators of the approved type are permitted in this examination. Candidates may also bring to the examination one double-sided, A4 sheet of notes.

Do not turn over until instructed.
Section A: Short Questions

A1. Sketch the graph of the function \( f(x) = \frac{1}{1+x^2} \). State the domain and the range of this function.

A2. Expand the quantity \((x+h)^4\). Use your result to find \( \lim_{h \to 0} \left\{ \frac{(x+h)^4 - x^4}{h} \right\} \).

A3. Which of these sequences converge: (i) \( \cos(n) \) and (ii) \( \tan^{-1}(n) \). Which of these series converge:

(i) \( \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \);  
(ii) \( \sum_{n=2}^{\infty} \frac{n}{n^2 - 1} \).

(State any methods used.)

A4. A complex number, \( z \), is written \( z = 2e^{i\pi/4} \) in complex exponential form. Express \( z \) in Cartesian form \( a + ib \). Also find \( z^3 \) and \( 1/z \) in either complex exponential or Cartesian form.

A5. (i) Differentiate \( \sin(\cos(x)) \) with respect to \( x \).

(ii) For what values of \( x \) is \( \ln \left| 1 - \frac{1}{x} \right| \) differentiable? Calculate the derivative when it is differentiable.

A6. Use the Leibniz formula to find the 5th derivative of \( (x+1) \sin x \).

A7. State de Moivre’s Theorem and use it to express \( \sin 3\theta \) in terms of \( \sin \theta \).

A8. Use partial fractions to evaluate \( \int \frac{1}{x(x^2 + 1)} \, dx \).

A9. Evaluate the following limits (if they exist) stating any methods used:

(i) \( \lim_{x \to 2} \left\{ \frac{x^3 - x^2 - x - 2}{x - 2} \right\} \);  
(ii) \( \lim_{x \to 0} \left\{ \frac{x + \tan x}{x - \tan x} \right\} \).

A10. Find the general solution, \( y(x) \), to the second order differential equation

\[ y'' + 2y' - 3y = x. \]
Section B: Longer Questions

B1. (a) (4 marks)
Sketch the graph of the function \( f(x) = (1-x)^{-1/2} \) and state its domain and range.

(b) (4 marks)
Calculate \( f'(x) \). Show, by induction, that the \( n \)th derivative of \( f(x) \) is
\[
f^{(n)}(x) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n(1-x)^{n+1/2}}.
\]

(c) (5 marks)
Write down the first 4 non-zero terms and the general term of the MacLaurin series expansion of \( f(x) \) about \( x = 0 \). Use your expansion to evaluate the infinite series
\[
1 + \frac{1}{2^2} + \frac{1.3}{2^42!} + \frac{1.3.5}{2^63!} + \ldots.
\]

(d) (2 marks)
Find the radius of convergence of the MacLaurin series in part (c).

B2. (a) (3 marks)
You are given the definition \( \cosh(x) = \frac{1}{2}(e^x + e^{-x}) \). Define \( \sinh(x) \) and \( \text{sech}(x) \) in terms of exponentials and use your definitions to confirm that
\[
(\cosh x + \sinh x)^n = \cosh nx + \sinh nx
\]
for any \( n \).

(b) (2 marks)
Sketch the curves of \( \cosh(x) \) and \( \text{sech}(x) \) on the same graph.

(c) (4 marks)
By using a suitable substitution, or otherwise, show that
\[
\int \text{sech } x \, dx = 2 \tan^{-1}(e^x) + \text{constant}
\]
and hence evaluate \( \int_{-\infty}^{\infty} \text{sech } x \, dx \).
(d) (3 marks)
Writing \( y = \cosh x \), \( x \geq 0 \) and using the definition in terms of exponentials, show that
\[
\cosh^{-1}(y) = \ln(y + \sqrt{y^2 - 1}).
\]

(e) (3 marks)
Calculate \( \frac{d}{dy} \cosh^{-1}(y) \).

B3. (a) (3 marks)
Does the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) converge? If not, explain why.

(b) (4 marks)
Determine for which values of \( x \) the following power series converge:

(i) \( \sum_{n=0}^{\infty} x^{2n} \);
(ii) \( \sum_{n=0}^{\infty} \frac{n^n x^n}{n!} \).

[For part (ii) you are given that \( \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \).]

(c) (5 marks)
Evaluate the series in (b)(i) when it converges, and use your result in the integral
\[
\int_{0}^{t} \frac{1}{1-x^2} \, dx,
\]
to show that, for \( |t| < 1 \),
\[
\tanh^{-1}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)}
\]

(d) (3 marks)
Using the result of part (c), or otherwise:
(i) evaluate \( \lim_{t \to 0} \left\{ \frac{\tanh^{-1}(t)}{t} \right\} \);
(ii) demonstrate that the series \( 1 + \frac{1}{3} + \frac{1}{5} \ldots \) does not converge.
B4. (a) (4 marks)

Find the following limits, in each case considering the limits from both above and below the limit point.

(i) \( \lim_{x \to 0} \frac{\sin x}{x} \); (ii) \( \lim_{x \to 0} \frac{\sin x}{|x|} \); (iii) \( \lim_{x \to 1} \sqrt{x - 1} \).

(b) (4 marks)

Define the modulus, \( |z| \) and complex conjugate \( \bar{z} \) of a complex number \( z = a + ib \) in terms of \( a \) and \( b \) and hence show that \( z\bar{z} = |z|^2 \). What relationship holds between \( a \) and \( b \) if \( z/\bar{z} \) is to be imaginary?

(c) (3 marks)

You are given the relation

\[ e^{i\theta} = \cos n\theta + i \sin n\theta. \]

Write down the equation which comes from taking the complex conjugate of both sides of the above. Hence express \( \cos n\theta \) and \( \sin n\theta \) in terms of complex exponentials.

(d) (4 marks)

Show that

\[ 1 + \cos \theta + \cos 2\theta + \ldots + \cos N\theta = \cos \left( \frac{1}{2} N\theta \right) \frac{\sin \left( \frac{1}{2} (N + 1)\theta \right)}{\sin \left( \frac{1}{2} \theta \right)}. \]

[You may use the result \( \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z} \) for complex variables \( z \).]

B5. (a) Consider the following differential equation for the function \( y(x) \):

\[ \frac{dy}{dx} + xy = x. \quad \text{with} \quad y(0) = 0. \]

(i) (2 marks)

Classify the differential equation in terms of its order, linearity and homogeneity.

(ii) (3 marks)

Find the solution to the differential equation.

Continued...
(b) In a model of a nuclear chain reaction, an initial mass \( M \) of matter \( A \) decays to form matter \( B \) which itself decays to form matter \( C \). The mass \( x(t) \) at time \( t \) of matter \( B \) is governed by the differential equation

\[
\frac{dx}{dt} = -px + qMe^{-qt}, \quad \text{with } x(0) = 0
\]

where \( p \) and \( q \) are positive rates of decay of \( B \) and \( A \) respectively.

(i) (4 marks)
Assuming \( p \neq q \) solve the differential equation to show

\[
x(t) = \frac{qM}{p-q} \left( e^{-qt} - e^{-pt} \right).
\]

(ii) (3 marks)
The mass of matter \( C \) after time \( T \) is given by

\[
M_C(T) = \int_0^T px(t) \, dt.
\]

Calculate \( M_C(T) \) and hence show that \( M_C(T) \to M \) as \( T \to \infty \).

(iii) (3 marks)
Show, by taking limits \( p \to q \) (or otherwise), that the solution for \( p = q \) is

\[
x(t) = qMte^{-qt}.
\]

Hence calculate \( M_C(T) \) and confirm again that \( M_C(T) \to M \) as \( T \to \infty \).

B6. (a) (3 marks)
Classify the following functions as: even/odd/neither and periodic/non-periodic.

(i) \( x^2 \); (ii) \( \sin^2 x \); (iii) \( x^2 \sin x \).

(b) This question concerns the function \( f(x) = x^2 \).

(i) (1 mark)
Sketch the graph of the function \( f \) for \(-4\pi < x < 4\pi \).

(ii) (2 marks)
You are given that the Fourier series for \( f \) on the interval \((-\pi, \pi)\) is

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).
\]

Write down integrals for \( a_n \) and \( b_n \) in terms of \( f(x) \).
(iii) (2 marks)
Sketch the graph of the Fourier series for \( f \) on the interval \([-\pi, \pi]\) over \(-4\pi < x < 4\pi\).

(iv) (4 marks)
Show that \( a_0 = \frac{2\pi^2}{3} \) and that \( b_n = 0 \) for all \( n = 1, 2, \ldots \).

(v) (3 marks)
You are given that \( a_n = 4(-1)^n/n^2 \). Use the Fourier series to show that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]