This paper contains two sections, Section A and Section B. Answer each section in a separate answer book.

Section A contains TEN short questions worth 4 marks each. All TEN answers will be used for assessment.

Section B contains SIX longer questions worth 15 marks each. A candidate’s FOUR best answers will be used for assessment.

Calculators of the approved type are permitted in this examination. Candidates may also bring to the examination one double-sided, A4 sheet of notes.
Section A: Short Questions

A1. Sketch the graph of the function \( \frac{x + 1}{x - 1} \) giving equations of any vertical and horizontal asymptotes and any points where the graph of the function crosses the axes.

Also sketch, in a separate figure, the graph of the function \( \frac{|x + 1|}{x - 1} \).

A2. An infinite sequence \( \{a_n\} \) is defined by the relation \( a_{n+1} = 3a_n - 2a_{n-1} \) for \( n \geq 1 \).

(a) Determine the two possible values of \( r \) for which \( a_n = Ar^n \) is a solution of the relation (\( A \) is an arbitrary constant.)

(b) For which of these initial values does the sequence \( \{a_n\} \) converge/diverge?

(i) \( a_0 = 1, \ a_1 = 1 \); (ii) \( a_0 = 0, \ a_1 = 1 \).

A3. Express the hyperbolic function \( \cosh(x) \) in terms of exponentials and use this definition along with the binomial expansion to confirm that

\[
\cosh^5(x) = \frac{1}{16} (\cosh(5x) + 5 \cosh(3x) + 10 \cosh(x)).
\]

A4. Consider the function \( f(x) = \ln(\ln(x)) \). What is the domain and range of the function \( f(x) \)? Calculate \( f'(x) \) and hence determine the value of

\[
\int_{e}^{e^2} \frac{1}{x \ln(x)} \, dx.
\]

A5. (a) Express the complex number \( i \) in complex exponential form. Hence find \( i^i \) in Cartesian form \( a + ib \).

(b) Using the fact that \( 2 = e^{\ln(2)} \), express \( 2^i \) in Cartesian form \( a + ib \). Also find \( |2^i| \).

A6. As shown in Question B2, the MacLaurin expansion of \( \sinh^{-1}(x) \) is

\[
\sum_{n=0}^{\infty} (-1)^n \frac{1^2 \cdot 3^2 \cdot 5^2 \cdots (2n - 1)^2}{(2n + 1)!} x^{2n+1}.
\]

Use the ratio test to determine the radius of convergence of this series.
A7. Express \( \frac{1}{n(n+2)} \) in terms of partial fractions. Hence evaluate \( \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \).

A8. Evaluate \( \int \frac{1}{x^2 + 4x + 20} \, dx \).

A9. Find the general solution \( y(x) \) to the separable differential equation,

\[
\frac{dy}{dx} = \frac{x^2 + 1}{xy}.
\]

A10. You are given that the Fourier series for \( f(x) \) on the interval \((-\pi, \pi)\) is

\[
\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).
\]

Evaluate \( b_n \) for all \( n \), and \( a_0 \) when \( f(x) = (x/\pi)^4 \).
Section B: Longer Questions

B1. (a) (2 marks)
Determine the following:

(i) \( \lim_{n \to \infty} \left\{ \frac{1}{n\pi + 1/(n\pi)} \right\} \); (ii) \( \lim_{n \to \infty} \left\{ \frac{1}{1 + 1/(n\pi)} \right\} \).

(b) (6 marks)
Deduce the first three terms (to include any that evaluate to zero) in the Taylor series expansion of the following functions about \( n\pi \) where \( n \) is a positive integer:

(i) \( \tan(x) \); (ii) \( \frac{1}{x} \).

(c) (i) (1 mark)
On the same graph sketch curves of \( \tan(x) \) and \( -1/x \) over the interval \( 0 \leq x \leq 4\pi \).

(ii) (3 marks)
Hence determine graphically that there is an infinite sequence of positive roots, \( \{x_1, x_2, \ldots\} \), of the equation \( x \tan(x) = -1 \) such that \( (n - \frac{1}{2})\pi < x_n < n\pi \).

(iii) (3 marks)
Letting \( \epsilon_n = x_n - n\pi \), show that \( \epsilon_n \approx \frac{1}{n\pi + 1/(n\pi)} \) as \( n \to \infty \).
B2. (a) (4 marks)
Sketch the graph of sinh(x). Explain why sinh x, where \( x \in (-\infty, \infty) \), has an inverse function sinh\(^{-1}\) and sketch its graph also.

(b) (4 marks)
Consider the function \( y(x) = \text{sinh}^{-1}(x) \). Show that

\[
y'(x) = \frac{1}{\sqrt{1 + x^2}}.
\]

Using this find \( y''(x) \) and hence show that

\[
(1 + x^2)y''(x) + xy'(x) = 0. \tag{1}
\]

(c) (3 marks)
Using the Leibniz formula take the \( n \)th derivative of (1) to show that, for \( n \geq 0 \),

\[
(1 + x^2)y^{(n+2)}(x) + (2n + 1)xy^{(n+1)}(x) + n^2y^{(n)}(x) = 0.
\]

(d) (5 marks)
Calculate \( y(0) \), \( y'(0) \) directly and hence use part (c) to show that \( y^{(n)}(0) = 0 \) for \( n \) even.

Also find a formula for \( y^{(n)}(0) \) when \( n \) is odd and hence deduce that the MacLaurin series of the function sinh\(^{-1}\)(x) is

\[
\sum_{n=0}^{\infty}(-1)^n \frac{1\cdot3\cdot5\cdots(2n-1)^2}{(2n+1)!}x^{2n+1}.
\]
B3. (a) (6 marks)
Find the following limits, if they exist:

(i) \( \lim_{x \to 0} \left\{ \frac{\sin 2x}{x} \right\} \); (ii) \( \lim_{x \to 0} \left\{ \frac{\cos 2x}{x} \right\} \); (iii) \( \lim_{x \to 0} \left\{ \frac{\cos(a + x) - \cos(a)}{x} \right\} \).

(a is a fixed constant).

(b) (2 marks)
Calculate the derivative of \([\ln(x)]^n\) for \(n\) a positive integer.

(c) (3 marks)
Show that \( \lim_{x \to 0} \{x[\ln(x)]^n\} = 0.\)

(d) Define \( I_n = \int_0^1 [\ln(x)]^n \, dx \).

(i) (2 marks)
Show that \( I_n = -nI_{n-1}, \quad \text{for } n \geq 1.\)

(ii) (2 marks)
Calculate \( I_0 \) and hence evaluate \( I_n.\)
B4. (a) (i) (2 marks)
For \( z = 1 + i \), find \( |z| \) and \( \arg(z) \) and hence express \( z \) in complex exponential form.

(ii) (4 marks)
Find all complex solutions of \( z^2 = (1 + i) \) and plot them on the Argand diagram.

(iii) (2 marks)
Hence find all complex solutions of the quadratic equation \( z^2 + 2z - i = 0 \).

(b) (2 marks)
If a complex number \( w = a + ib \) where \( a \) and \( b \) are real, define the complex conjugate \( \bar{w} \) and confirm that \( |w|^2 = w\bar{w} \).

(c) In physics, the complex transmission coefficient, \( w \), for wave propagation through a slit diffraction grating satisfies the relation \( |w|^2 + |1 - w|^2 = 1 \).

(i) (3 marks)
Show that \( |w| = \cos \theta \) where \( \theta \) is the argument of \( w \).

(ii) (2 marks)
Hence show that \( w = \frac{1}{2} + \frac{1}{2}e^{2i\theta} \) and interpret this solution geometrically.
B5. (a) (3 marks)
Find the general solution to the differential equation for the function $y(x)$:

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1} y = 1.$$ 

(b) (3 marks)
Apply the following conditions to your general solution from part (a) and seek unique solutions in each case:

(i) $y(0) = 1$; (ii) $y(1) = 0$.

(c) A planet of radius $R$ has a shaft passing through its centre which connects one side of the planet to the other. The gravitational acceleration on the surface of the planet is denoted by $g$.

The idea is to jump into the shaft on one side of the planet and emerge some time, $T$, later on the other side. Your motion is determined by the equation

$$\frac{d^2x}{dt^2} = -\frac{g}{R} x$$

for the unknown $x(t)$ with initial conditions $x(0) = R$ and $\frac{dx(0)}{dt} = 0$.

(i) (6 marks)
Solve the differential equation, apply the initial conditions and hence show that $x(T) = -R$ is satisfied by $T = \pi \sqrt{R/g}$.

(ii) (1 mark)
On Earth, $g \approx 10\text{ms}^{-2}$, $R \approx 6,400,000\text{m}$. Use these values to calculate $T$ to the nearest minute (if you don’t have a calculator, use $\pi \approx 3$ to estimate $T$).

(iii) (2 marks)
Newton’s law of gravitation states that $g = GM/R^2$ where $G$ is a constant and $M = \frac{4}{3} \pi \rho R^3$ in terms of the density, $\rho$, of the planet. How does the value of $T$ depend on the radius of the planet?
B6. (a) (3 marks)
Classify each of the following functions as even/odd/neither and periodic/non-periodic:

(i) \( \sin x \);  
(ii) \( |\sin x| \);  
(iii) \( \sin |x| \).

(b) This question concerns the function \( f(x) = \sin |x| \).

(i) (2 marks)
Sketch the graph of the function in the interval \(-\pi < x < \pi\). Is \( f(x) \) continuous?

(ii) (2 marks)
Calculate \( f'(x) \). Is \( f'(x) \) continuous?

(c) The Fourier series of a function is given in Question A10.

(i) (6 marks)
Calculate the Fourier coefficients \( a_n \) and \( b_n \) for the function \( f(x) = \sin |x| \) and hence show that the Fourier series of this function is

\[
\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.
\]

(ii) (2 marks)
Use the result of (c)(i) to show that

\[
\frac{\pi - 2}{4} = \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \ldots.
\]

End of examination.