Differentiation. Taylor Polynomials

Easy Questions

1. (Homework, parts (c), (d)) Differentiate the following, stating for what values of the variable the result holds:
   (a) \((1 - x^2)^2\); (b) \(\sin^2 t\); (c) \(\sin^2 t^2\); (d) \(\theta^2 \tan \theta\); (e) \(\sqrt{u^3 + 1}\).

2. (Homework, parts (a), (d)) Differentiate the following, stating for what values of the variable the result holds:
   (a) \(\cos^{-1} x\); (b) \(e^{ax} \cos(bt)\), where \(a\) and \(b\) are constants; (c) \(e^{x^2 - \sin x}\);
   (d) \(z^2\) (hint: see section 4.2.1 of the notes); (e) \(\tan^{-1}(\tan \theta)\); (f) \(\tan^{-1}(\cos \theta)\).

3. (Homework) Find \(\frac{d}{du} \tanh^{-1}(u)\).

4. Use the Leibniz formula to find the 6th derivative of \(x \sinh x\).

5. (Homework) Find: (i) the 5th derivative; and (ii) the \(n\)th derivative of \((x^2 + 1)e^{2x}\).

6. Find the Taylor polynomial of the given order and about the given point for each of the following functions:
   (a) \(\sin x\) order 3, about \(x = 0\);
   (b) \(\sin x\) order 4, about \(x = 0\).
   Comment on your answers for (a) and (b).
   (c) \(x^3\) order 2 about \(x = 0\). Comment on this result.

Standard Questions

7. It is tempting to think the following statement about functions is true: “if \(f(x) \to 0\) as \(x \to \infty\) then \(f'(x) \to 0\) as \(x \to \infty\) also”.

   Disprove this statement using the function \(f(x) = \frac{\sin(x^2)}{x}\). Can you find a function such that both \(f(x) \to 0\) and \(f'(x) \to 0\) but \(f''(x) \not\to 0\) as \(x \to \infty\) ?

8. (a) Sketch the graph of \(f(x) = x|x|\). Just using the graph, answer the question “is \(x|x|\) differentiable everywhere?”.

   (b) Evaluate \(f'\) for \(x > 0\) and for \(x < 0\), by using the fact that \(|x| = x\) and \(-x\) in these two cases. Express the result in a single formula holding for all \(x \neq 0\).

   (c) Show that \(\frac{f(h) - f(0)}{h}\) tends to a limit as \(h \to 0\). Hence write down the value of \(f'(0)\). Finally, give an expression for \(f'\) that holds for all \(x\).

9. (Homework)

   (a) Calculate the 1st and 2nd derivatives of the function \(\frac{1}{1 - x}\). Infer a general expression for the \(n\)th derivative of this function.

   (b) Use the Leibniz formula and part (a) to calculate the \(n\)th derivative of \(\frac{x}{1 - x}\).

   (c) You should find the answers to (a) and (b) are the same. Why?

10. If \(x\) and \(y\) are related by \(y = f(x)\) and \(x = f^{-1}(y)\) and \(f'(x) \neq 0\) for all \(x\), show that

    \[
    \frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}
    \]

    [Hint: write \(p(x) = dy/dx\) and use the fact that \(dx/dy = 1/(dy/dx) = 1/p\).]

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11. (Homework)
Suppose you’ve lost your scientific calculator and you only have a basic calculator to hand with no scientific functions. How could you use it to find an approximation to $\tan(3)$ using a first degree Taylor polynomial? What answer would you get? How accurate would it be?

[Hint: 3 is not too far from $\pi$.]

12. A cable hangs under gravity between two pylons. The laws of mechanics say that its vertical sag $y$ as a function of horizontal distance $x$ satisfies the equation

$$\frac{d^2y}{dx^2} = \frac{1}{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where $a = T/mg$ and $m$ is the mass per unit length of the cable, $T$ is its tension, and $g$ the acceleration of gravity.

(a) Show, by substitution, that $y = C + a \cosh((x - D)/a)$ is a solution of that equation for any constants $C$ and $D$.

(b) For simplicity suppose the two pylons are of equal height and placed at $x = \pm L$ – this allows us to set $D = 0$.

Assume that the cable is light and under high tension in such a way that $L/a$ is small. Show, using Taylor polynomials, that the shape of the cable is approximately parabolic.

——— Harder Questions ———

13. Define $f(x) = \begin{cases} x \sin(1/x), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0 \end{cases}$, and $g(x) = xf(x)$ for all $x$.

Are $f$ and $g$ continuous at 0? Sketch their graphs. Are they differentiable at 0?

14. If $f$ is differentiable at $a$, evaluate $\lim_{x \to a} \left( \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \right)$.

15. The hydrogen atom is described in quantum mechanics by its “wave function” $\psi$ ($\psi$ is the Greek letter psi). The wave functions for the ‘$m = 0$’ states of the atom are given by $\psi = R(r)P_l(\cos \theta)$. Here $(r, \theta)$ are coordinates in 3D space The number $l$ is an integer, one of the “quantum numbers” of the atom. We focus on the function $P_l$ which is called the Legendre polynomial of degree $l$. It is defined by

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$  \hspace{1cm} (1)

(a) Show that $P_0(x) = 1$, $P_1(x) = x$, and find $P_2(x)$.

(b) Show that $P_l(x)$ is an even or odd function according to whether $l$ is even or odd.

(c) [Really really hard] Can you show

$$(l + 1)P_{l+1}(x) = (2l + 1)xP_l(x) - lP_{l-1}(x)$$

(d) Use this formula to deduce $P_3(x)$ and $P_4(x)$ starting from your answer to (a).