Basic Algebra and Trigonometry

1. (a) $(x - 1)(x + 1)$;
   (b) $(a - 2b)^2$;
   (c) vanishes at $t = 1$; hence $(t - 1)$ is a factor. Factorising gives $t^3 - 7t + 6 = (t - 1)(At^2 + Bt + C)$.

2. (a) Factorise as $x(x + 2) = 0$ so $x = 0$ or $x = -2$;
   (b) Use quadratic roots formula: $y = \frac{1}{2}(1 \pm \sqrt{5})$;
   (c) Use formula to get two identical roots: $a = \frac{3}{2}$.

3. Straight from the binomial expansion:
   (a) $(1 + 1/t)^4 = 1 + 4/t + 6/t^2 + 4/t^3 + 1/t^4$;
   (b) $(2 + x)^5 = 2^5 + 5 \cdot 2^4 x + 10 \cdot 2^3 x^2 + 10 \cdot 2^2 x^3 + 5 \cdot 2 x^4 + x^5$.

4. Standard results:
   (a) $1/\sqrt{2}$;
   (b) $\sqrt{3}/2$;
   (c) $\tan(19\pi/4) = \tan(4\pi + 3\pi/4) = \tan(3\pi/4) = \tan(\pi - \pi/4) = -\tan(\pi/4) = -1$;
   (d) $1/\cos(2\pi + \pi) = 1/\cos(\pi) = -1$.

5. (a) This is an easy quartic, because there are no odd powers of $x$, so it’s just a quadratic equation in $x^2$. So from the formula, $x^2 = \frac{1}{2}(3\pm \sqrt{3} - 8) = 1$ or 2. Hence $x = \pm 1$ or $\pm \sqrt{2}$.
   (b) Following the method of 1(c), look for simple factors by looking for values of $x$ which make the left hand side 0. Obviously $x = 1$ works, so $(x - 1)$ is a factor, and we have

$$x^3 + 6x^2 - 8x + 1 = (x - 1)(Ax^2 + Bx + C)$$

Equating coefficients, as in 1(c) we find $A = 1$, $B = 7$, $C = -1$. Hence,

$$x^3 + 6x^2 - 8x + 1 = (x - 1)(x^2 + 7x - 1)$$

and the (three) roots are $x = 1$ or $x = \frac{1}{2}(-7 \pm \sqrt{53})$.

6. From the definition of the binomial coefficient

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!}{(k+1)!(n-k-1)!}((k+1) + (n-k))$$

$$= \frac{n!}{(k+1)!((n+1) - (k+1))!} = \left(\frac{n+1}{k+1}\right)$$

7. (a) It’s helpful to look at the graph.

Two solutions are $\pi/6$ and $\pi - \pi/6 = 5\pi/6$. These are the only solutions between 0 and $2\pi$; the full set of solutions is $\pi/6 + 2n\pi$ and $5\pi/6 + 2n\pi$ for all integer values of $n$ (that is, $n = ..., -2, -1, 0, 1, 2, ...$).

(b) Two methods, the first being the better.

Method 1: Use a trigonometric identity. So

$$0 = \sin(7\theta) - \sin(5\theta) = \sin(6\theta + \theta) - \sin(6\theta - \theta) = \sin 6\theta \cos \theta + \cos 6\theta \sin \theta - \sin 6\theta \cos \theta + \cos 6\theta \sin \theta = 2 \sin \theta \cos 6\theta$$

Hence

- either $\sin \theta = 0$ giving $\theta = n\pi$ for integer $n$,
- or $\cos(6\theta) = 0$, giving $6\theta = (n + \frac{1}{2})\pi$, hence $\theta = (2n + 1)\pi/12$.

Method 2: Use the fact that if $\sin x = \sin y$, then either $x = y + 2n\pi$ or $x = \pi - y + 2n\pi$ for integer $n$. This follows from the graph of $\sin x$ as in part (a). So, we have:

- either $7\theta = 5\theta + 2n\pi$, giving $2\theta = 2n\pi$,
- or $7\theta = \pi - 5\theta + 2n\pi$ giving $12\theta = (2n + 1)\pi$.

So the solutions are $n\pi$ and $(2n + 1)\pi/12$ for integer $n$.

The answers agree with the first method.

(c) Standard identity: $\cos 2\theta = 2\cos^2 \theta - 1$. So the eqn. is $\cos 2\theta + 1 = 1$, so $\cos 2\theta = 0$, $2\theta = (n + \frac{1}{2})\pi$ or $\theta = (2n + 1)\pi/4$ for integer $n$.

(d) We have

$$2\sin \theta \cos \theta = 1 + 2\cos^2 \theta - 1 = 2\cos^2 \theta$$

That is, $\cos \theta (\cos \theta - \sin \theta) = 0$.

So:
• either \( \cos \theta = 0 \), giving \( \theta = (n + \frac{1}{2})\pi \).
• or \( \sin \theta = \cos \theta \). Solutions of this last equation are the same as solutions of \( \tan \theta = 1 \), which is satisfied by \( \theta = (n + \frac{1}{2})\pi \).

So the full set of solutions are \( (n + \frac{1}{2})\pi, (n + \frac{1}{2})\pi \) for \( n \in \mathbb{Z} \).

8. (a) Let D be the foot of the perpendicular from B to side b. Then \( \triangle ABD \) gives \( c^2 = (b - a \cos C)^2 + (a \sin C)^2 \).

The cosine rule follows immediately.

(b) Using the same diagram as in (a), BD = a \sin C from triangle BCD, but also BD = c \sin A from triangle ABD. Hence \( a \sin C = c \sin A \), and dividing through gives \( a/\sin A = c/\sin C \). Since there’s complete symmetry between A,B,C, it follows that \( b/\sin B = \) the others.

9. Let \( a = c \cos \phi \) and \( b = c \sin \phi \). In other words, \( c = \sqrt{a^2 + b^2} \) and \( \phi = \tan^{-1}(b/a) \). Then the equation to solve is

\[
c \sin \theta \sin \phi + c \cos \theta \cos \phi = 1
\]

using double angles, \( \cos(\theta - \phi) = 1/c \) so \( \theta = \phi + \cos^{-1}(1/c) \) or

\[
\theta = \tan^{-1}(b/a) + \cos^{-1}(1/\sqrt{a^2 + b^2})
\]

10. Standard identities used here:

\[
\sin 3\theta = \sin(\theta + 2\theta) = \sin \theta \cos 2\theta + \cos \theta \sin 2\theta = \sin \theta(1 - 2 \sin^2 \theta) + \cos \theta, 2 \sin \theta \cos \theta
\]

Finally use \( \cos^2 \theta = 1 - \sin^2 \theta \) to reduce this to the given formula.

11. Start with the formula

\[
(1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n} x^n
\]

and simply substitute \( x = 1 \) to get the result.

12. Here the idea is that we plot \( y = \tan x \) and \( y = \sqrt{a^2 - x^2} \) on the same graph. Wherever the curves intersect, they have the same value of \( y \) and the corresponding \( x \) is a root of the given equation.

The curve \( y = \sqrt{a^2 - x^2} \) for \( x > 0 \) is the positive circular segment of radius \( a \). From the graphs it can be seen that

- there is 1 root if \( a < \pi \);
- 2 roots if \( \pi \leq a < 2\pi \), ...

and generalising this, \( n \) roots if \( (n - 1)\pi \leq a < n\pi \).

13. Write \( \pi/12 = \pi/3 - \pi/4 \).

Then \( \sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2} \), \( \sin(\pi/3) = \sqrt{3}/2 \), \( \cos(\pi/3) = 1/2 \).

Then \( \sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4) = (\sqrt{3}/2)(1/\sqrt{2}) - (1/2)(1/\sqrt{2}) = (\sqrt{3} - 1)/2\sqrt{2} \).

Similarly \( \cos(\pi/12) = \cos(\pi/3 - \pi/4) = \cos(\pi/3)\cos(\pi/4) + \sin(\pi/3)\sin(\pi/4) = (1/2)(1/\sqrt{2}) + (\sqrt{3}/2)(1/\sqrt{2}) = (\sqrt{3} + 1)/2\sqrt{2} \).

(Note that there are many other ways of expressing these numbers.)

14. (a) Let \( r, s \) be the two given rational numbers with \( r < s \). Take a prime number \( p \) such that \( \sqrt{p} > 1/(s - r) \). Then \( s > r + 1/\sqrt{p} \) and so \( r + 1/\sqrt{p} \) is an irrational number which lies between \( r \) and \( s \).

(b) Follows from the idea that if you truncate the infinite decimal expansion of an irrational number \( x \), keeping only finitely many decimal places, you get a rational number which is as close to \( x \) as you like: the more decimal places you keep, the closer to \( x \) is the rational number you get.