Maths 1A20 Calculus 2014-15
Solution Sheet 5

Differentiation. Taylor Polynomials

1. (a) \(2(1 - x^2)(-2x)\). Holds for all \(x\);
(b) \(2 \sin t \cos t\). Holds for all \(t\);
(c) \(2 \sin^2 t \cos^2 t(2t)\). Holds for all \(t\);
(d) \(2 \theta \tan \theta + \theta^2 \sec^2 \theta\) provided \(\theta \neq (n + \frac{1}{2})\pi\) for some integer \(n\);
(e) \(\frac{1}{2}u^2/\sqrt{u^4 + 1}\) provided \(u > -1\). Note that the given function has domain for \(u \geq -1\) but it is not differentiable at \(u = -1\).

2. (a) \(-1/\sqrt{1 - x^2}\) for \(|x| < 1\). Note that the \(\cos^{-1}\) function is defined for \(|x| \leq 1\) but it is not differentiable at \(\pm 1\), and in fact does not even have one-sided derivatives there;
(b) \([a \cos \theta - b \sin \theta]e^{\alpha t}\). Holds for all \(t\);
(c) \(e^{x^2-\sin^2(2s - \cos s)}\). Holds for all \(s\);
(d) Following the hint in the question leads to \(z^2 = e^{x \ln z}\). Differentiating gives
\[
\frac{d}{dz}(z^2) = \frac{dz}{dz}e^{x \ln z} = e^{x \ln z}(\ln z + z/\ln z) = (1 + \ln z)z^2
\]
Holds for \(z > 0\) (otherwise the logarithm doesn’t exist).
(e) For \(\theta \neq (n + \frac{1}{2})\pi\) we have \(\tan^{-1}(\tan \theta) = \theta + n\pi\) where \(n\) is an integer such that \(\theta + n\pi\) is in \((-\pi/2, \pi/2)\). The derivative is 1 everywhere except \((n + \frac{1}{2})\pi, n\) an integer; these points are outside the domain of the function, since \(\tan \theta\) doesn’t exist at \(\tan(n + \frac{1}{2})\pi\).
(f) Chain rule gives \(-\sin \theta / (1 + \cos^2 \theta)\) Valid for all \(\theta\).

3. Let \(v = \tan^{-1} u\), then \(u = \tan v\) so \(du/dv = \sec^2 v\). Therefore \(dv/du = 1/\sec^2 v\). We must express this in terms of \(u\). Since \(-1 - \tan^2 u = \sec^2 u\), we have
\[
\frac{d}{du} \tan^{-1} u = \frac{1}{\sec^2 v} = \frac{1}{1 - \tan^2 u} = \frac{1}{1 - u^2}
\]
This is the answer.

4. Don’t differentiate 6 times. If \(g(x) = x\) and \(f(x) = \sin x\) then Leibniz gives \(f^{(6)}(x)g(x) + 6g'(x)f^{(5)}(x)\) since \(g^{(n)}(x)\) and higher derivatives are zero. So the answer is
\[x \sin x + 6 \cos x - 2x \sin x\]

5. (i) Don’t differentiate 5 times. Use the Leibniz rule, and the fact that the \(n\)-th derivative of \(e^{ax}\) is \(a^n e^{ax}\). Then we have:
\[
\frac{d^n}{dx^n}(e^{2x}(x^2 + 1)) = 2^n e^{2x}(2x^2 + 1) + 5.2^4 e^{2x}(2x) + 10.2^6 e^{2x}(2) + 0 = (32x^2 + 160x + 192)e^{2x}
\]
(ii) Now the general formula, done the same way
\[
\frac{d^n}{dx^n}(e^{2x}(x^2 + 1)) = 2^n e^{2x}(2x^2 + 1) + n(2^n-1)e^{2x}(2x)
\]
\[
+ \frac{n(n-1)}{2}(2^{n-2}e^{2x})(2) + 0
\]
\[
= 2^n(e^{2x} + nx + 1 + \frac{1}{2}(n-1))e^{2x}
\]

6. (a) Write \(f(x) = \sin x\), then \(f(0) = \sin 0 = 0\), \(f'(0) = \cos (0) = 1\), \(f''(0) = -\sin 0 = 0\), \(f'''(0) = -\cos 0 = -1\). Hence the Taylor Polynomial of order 3 for \(\sin x\) is \(x - \frac{1}{2}x^3\);
(b) Again \(f(x) = \sin x\) and we’ve done up to third derivative in part (a). Continuing then \(f(4)(0) = \sin 0 = 0\), so the 4th order Taylor Polynomial is the same as the 3rd order Taylor Polynomial.

This means that the cubic approximation cannot be improved by adding an \(x^4\) term. (Of course, it can be improved by adding an \(x^5\) term).
(c) Write \(f(x) = x^3\), then \(f(0) = f'(0) = f''(0) = 0\). Hence the 2nd order Taylor polynomial is just 0.

The point here is that for \(x\) near 0, \(x^3\) is much smaller than \(x^2\). So it is impossible to approximate \(x^3\) satisfactorily using lower powers; 0 is a better approximation to \(x^3\) than any other combination of \(x^3, x, x^2\).

7. Here, \(\sin(x^2)\) is bounded between 1 and -1 so \(f(x) = \sin(x^2)/x \to 0\) as \(x \to \infty\). But
\[
f'(x) = -\frac{\sin x^2}{x^2} + 2 \cos x^2
\]
and whilst the first term tends to zero, the second doesn’t. How does this happen ? As the function decays it simultaneously oscillates more and more rapidly so that the derivative is non-vanishing.

For the next part, try \(f(x) = \sin(x^{3/2})/x\). Find \(f'(x)\) decays but \(f''(x) \sim -\frac{9}{2} \sin(x^{3/2})\) as \(x \to \infty\) and doesn’t.

8. (a) The graph looks like \(x^2\) for \(x > 0\) and \(-x^2\) for \(x < 0\), so gradients from the right and left of 0 both approach the same value of zero. In short, the curve looks smooth everywhere. So it looks differentiable. (b) For \(x > 0\) we have \(f(x) = x^2\) so \(f'(x) = 2x\). For \(x < 0\) we have \(f(x) = -x^2\) so \(f'(x) = -2x\).

The two cases of \(x > 0\) and \(x < 0\) can be accounted for in the formula \(f'(x) = 2|x|\) for \(x \neq 0\).
(c) \(\frac{f(h) - f(0)}{h} = \frac{h|h| - 0}{h} = |h|\)

This \(\to 0\) as \(h \to 0\) (i.e. from both sides of zero).

This means \(f\) is differentiable at 0 (and hence everywhere, as we suspected from part (a)), and \(f'(0) = 0\).

So \(f'(x) = 2|x|\) holds for all \(x\), not just for \(x \neq 0\).

9. (a) If \(f(x) = 1/(1 - x)\) then \(f'(x) = 1/(1 - x)^2\) and second is \(f''(x) = 2/(1 - x)^4\). Can infer that the \(n\)th derivative is
\[
f^{(n)} = \frac{n!}{(1 - x)^{n+1}}
\]

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(b) If we let $g(x) = x$ then $g'(x) = 1$ and all other
derivs are zero. So Leibniz gives

$$
\frac{d^n}{dx^n} \left( \frac{x}{1-x} \right) = x \frac{n!}{(1-x)^{n+1}} + n \frac{(n-1)!}{(1-x)^n} = \frac{n!}{(1-x)^{n+1}}
$$

(c) Because $\frac{x}{1-x} = \frac{1}{1-x} - 1$, the derivatives of
$x/(1-x)$ are the same as those for $1/(1-x)$.

10. This stems from the fact that, in the notes, we say
that $dy/dx = 1/(dx/dy)$. We also say that the form-
ula does not generalise in such an obvious way to higher
derivatives. This question is designed to show the
relation that holds between second derivatives
d$^2y/dx^2$ and $d^2x/dy^2$. Here goes:

$$
\frac{d^2x}{dy^2} = \frac{d}{dy} \frac{dx}{dy} = \frac{d}{dy} \left( \frac{1}{p} \right) = \left( \frac{-1}{p^2} \right) \frac{dp}{dy}
$$

$$
= \left( \frac{-1}{p^2} \right) \frac{dx}{dy} \frac{dp}{dx} = \left( \frac{-1}{p^2} \right) \left( \frac{1}{p} \right) \frac{d^2y}{dx^2}
$$

$$
= \frac{-d^2y}{ax^2} / \left( \frac{dy}{dx} \right)^3
$$

after reinstating $p = dy/dx$.

11. 3 is quite close to $\pi$, so use the first order Taylor
polynomial about $\pi$. The Taylor Polynomial is $\tan(\pi + (x - \pi) \sec^2 \pi$, since the derivative of tan is sec$^2$.
Since cos $\pi = -1$ and sec$^2 \pi = 1$ whilst $\tan \pi = 0$ so we have

$$
tan 3 \approx (3 - \pi) \approx -0.142
$$
to 3 decimal places. (There’s no point in keeping a
large number of decimal places, considering how crude
an approximation we are making here.)

In fact $tan 3 = -0.1425\ldots$, so this crude approxima-
tion is not bad.

12. (a) $\frac{dy}{dx} = \sinh((x - D)/a); \frac{d^2y}{dx^2} = a \cosh((x - D)/a)$.

Now just plug into the given equation and use
$
\cosh t - \sinh t = 1.
$

(b) We are told $D = 0$ so $y = C + a \cosh(x/a)$ for
$-L \leq x \leq L$. If $L/a$ is small then $x/a$ is small for
all $x$ in $[-L, L]$. The 2nd degree Taylor polynomial
of $\cosh(x/a)$ is

$$
1 + \frac{x^2}{2a^2}
$$

If you take 1st degree Taylor polynomial, then cosh is
approximated by 1 which is not enough. If you take
the 4th degree then there is an extra term of $x^4/(4!a^4)$
but this is much smaller than $x^2/(2a^2)$ for $x/a$ small.
In summary then

$$
y \approx C + a + \frac{x^2}{2a}
$$

which is a parabola.

**Interesting stuff:** The curve formed by a hang-
ing chain is called a “catenary” (it’s actually just the
function “cosh”). You can look up its deriv-
ation and history on wikipedia (it attracted the inter-
est of famous mathematicians such as Galileo, Hooke,
Bernaulli, Huygens, Leibniz)

13. Clearly $f(x) \rightarrow 0$ as $x \rightarrow 0$ since $|f(x)| \leq x$. Since
$f(0) = 0$, it follows that $f$ is continuous at 0.

Even though the function behaves pretty strangely
near $x = 0$, it is still continuous.

$g$ is also continuous for the same reason.

For $x \neq 0$ we have $f'(x) = \sin(1/x) + x(-1/x^2)\cos(1/x)$
which diverges as $x \rightarrow 0$. The wig-
gles in the graph get steeper and steeper as $x \rightarrow 0$.

In fact $f$ is not differentiable at 0 because
$\frac{f(h) - f(0)}{h} = \sin(1/h)$ does not converge as $h \rightarrow 0$.

Similarly $g'(x) = 2x\sin(1/x) - \cos(1/x)$
for $x \neq 0$, which oscillates and does not converge as $x \rightarrow 0$.

But in fact $g$ is differentiable at 0, because

$$
g(h) - g(0) \over h = \sin(1/h) \rightarrow 0, \quad \text{as } h \rightarrow 0
$$

This example shows that even if $g'(x)$ behaves badly
as $x \rightarrow a$, $g'(a)$ may yet exist.

14. Multiply top and bottom by $\sqrt{x} + \sqrt{a}$, giving

$$
\lim_{x \rightarrow a} \left( \frac{\sqrt{x} + \sqrt{a}}{x-a} \right) \left( \frac{f(x) - f(a)}{x-a} \right) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}.
$$

But $\sqrt{x} + \sqrt{a} \rightarrow 2\sqrt{a}$
as $x \rightarrow a$, and the rest of the stuff inside the limit
$\rightarrow f'(a)$. Hence the limit is $2\sqrt{a}f'(a)$.

15. (a) 2$^9 = 1$ and 0$^1 = 1$, so $P_0$ is just $(x^2 - 1)^0 = 1$.

Next, $P_1(x) = \frac{1}{2} \frac{d}{dx} (x^2 - 1) = x$.

Then

$$
P_2(x) = \frac{1}{4} \frac{d^2}{dx^2} (x^4 - 2x^2 + 1) = \frac{1}{8} (12x^2 - 4) = \frac{1}{2} (3x^2 - 1)
$$

(b) NOTE: we talk about even numbers $n$, even powers,
and even functions. These are different (though related)
uses of the word “even” - potentially confusing if you
don’t think carefully.

If $n$ is even, $P_n(x)$ is an even-order derivative of a
combination of even powers of $x$, giving again only even
powers. So $P_n(x)$ is an even function.

If $n$ is odd, $P_n(x)$ is an odd-order derivative of a
combination of even powers, giving only odd powers of $x$;
so $P_n(x)$ is odd.

(c) One method is to get an explicit formula for $P_1$
by expanding $(x^2 - 1)^l$ using the binomial expansion
in (1) and then differentiating. Then plug into the
recursion formula of (15(c) and check that it is satisfied.
The details are very complicated; this is a hard
question.

There are other approaches to deriving the relation
in 11(c), involving differential equations.

(d) Take $l = 2$ giving

$$
3P_3 = 5xP_2 - 2P_1 = \frac{5}{2} x^3 - 2x = \frac{15}{2} x^3 - \frac{9}{2}
$$

Hence $P_3(x) = \frac{1}{2} (5x^3 - 3x)$.

Similarly

$$
4P_4 = 7xP_3 - 3P_2 = \frac{7}{2} (5x^3 - 3x) - \frac{3}{2} (3x^2 - 1)
$$

so $P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3)$.