UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sc. (Levels 3 and M)

INTRODUCTION TO QUEUEING NETWORKS
MATH 35800/M5800
(Paper Code MATH-35800/M5800)

April 2013, 1 hours and 30 minutes

This paper contains three questions
A candidate’s TWO best answers will be used for assessment.
Calculators are not permitted in this examination.

Do not turn over until instructed.
1. (a) (15 marks) Suppose $X_1, X_2, \ldots, X_k$ are independent Poisson random variables, with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. Let $Y = X_1 + X_2 + \ldots + X_k$. Using generating functions or otherwise, show that $Y$ is a Poisson random variable, with parameter $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_k$.

(b) (15 marks) Suppose $N$ is a Poisson random variable with parameter $\lambda$, and that $Y$ is Binomial with parameters $(N, p)$. To put it another way, suppose $X_1, X_2, \ldots$ are independent and identically distributed (iid) Bernoulli($p$) random variables, independent of $N$, and let $Y = \sum_{j=0}^{N} X_i$ (with the convention that an empty sum is zero). Using generating functions or otherwise, show that $Y$ is Poisson distributed, with parameter $\lambda p$.

(c) (20 marks) The number of patients admitted to a certain hospital ward each day can be modelled as a Poisson random variable. The mean number admitted each day is 12, and numbers admitted on different days are mutually independent random variables. Each admitted patient is discharged the next day with probability $1/4$, two days later with probability $1/2$, and three days later with probability $1/4$, independent of other patients, and of the arrival process.

Let $N_t$ denote the number of patients admitted to the ward on day $t$, and $Q_t$ the number of patients remaining in the ward at the end of day $t$ (which includes all those admitted that day, and excludes all those discharged that day). Using the answers to parts (a) and (b), show that $Q_t$ is a Poisson random variable, and compute its mean. If the ward is required to provide a number of beds which is at least three standard deviations above the mean occupancy, how many beds should it provide? You may leave your answer in a form involving square roots.
2. Consider a system of two parallel $\cdot/M/\infty$ queues, with external arrivals occurring according to a Poisson process of rate $\lambda$. If an arrival finds $n_1$ customers in the first queue and $n_2$ customers in the second, then it joins the first queue with probability $\frac{n_1+1}{n_1+n_2+2}$ and the second queue with probability $\frac{n_2+1}{n_1+n_2+2}$.

The service rate in both queues is $\mu$. In other words, each customer spends a random time in the system which is exponentially distributed with parameter $\mu$, and is independent of the arrival process, the routing choices, and the times spent by other customers in the system.

(a) (15 marks) Let $N(t) = (N_1(t), N_2(t))$ denote the number of customers in the two queues at time $t$, and note that $N(t)$ is a Markov process. What is the state space? Write down the transition rates for all possible transitions.

(b) (15 marks) Show that the Markov process $N(t)$ has invariant distribution

$$\pi(n_1, n_2) = c \frac{\rho^{n_1+n_2}}{(n_1 + n_2 + 1)!},$$

where $\rho = \lambda/\mu$ and $c > 0$ is a suitable constant. (You don’t need to show there is such a constant in this part of the question.)

(c) (20 marks) Find the constant $c$ in the expression for the invariant distribution in (1), stating any condition that $\rho$ must satisfy for there to be an invariant distribution.

Hint. Note that the number of states having the same value $k$ of $n_1 + n_2$ is $k + 1$, and that all these states have the same invariant probability.

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Continued...
3. A fairground has two rides, named Scary and Terrifying. People arrive at the fairground individually, according to a Poisson process of rate 3 per minute. On arrival, 60% of them go to the Scary ride and 40% to the Terrifying one. Each ride has its own ticket booth, manned by a single server. The server at each booth sells tickets to customers in their order of arrival at the booth, taking an exponentially distributed time with mean 20 seconds to deal with each customer. The times to deal with different customers are mutually independent, and independent of any personal characteristics or of the past evolution of the system.

The time spent on the Scary ride is uniformly distributed between 4 and 6 minutes, while that spent on the Terrifying ride is uniformly distributed between 6 and 10 minutes. The rides may be assumed to have infinite capacity, so all waiting customers can use them in parallel.

Customers who buy a ticket for a ride always go on it. Each person who has just been on the Scary ride next goes to the Terrifying ride with probability 0.4, and leaves with probability 0.6, independent of how many times they have been on either ride before, or of anything else in the past. Likewise, each person who has just been on the Terrifying ride next goes to the Scary ride with probability 0.1, and leaves with probability 0.9. A separate ticket has to be bought each time you want to go on a ride, and we assume that nobody buys multiple tickets.

(a) **(10 marks)** Model the number of people in the fairground at any time, and the rides or ticket booths where they are located at that time, as a queueing network. For each node in the network, specify what type of queue it is, the external arrival rate, and the service rate and service discipline at that queue. Also specify the routing probabilities between nodes.

(b) **(10 marks)** Write down and solve the traffic equations describing the total rate at which customers pass through each of the queues.

(c) **(10 marks)** What is the mean length of time spent in the fairground by a typical customer?

(d) **(10 marks)** If a ticket for the Terrifying ride costs £1, then how many pounds per hour does it earn the fairground operator, on average?

(e) **(10 marks)** What is the mean number of visits to the Terrifying ride by a typical customer coming to the fairground?

*End of examination.*