

LETTER TO THE EDITOR

Nodal domain distributions for quantum maps

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Abstract. The statistics of the nodal lines and nodal domains of the eigenfunctions of quantum billiards have recently been observed to be fingerprints of the chaoticity of the underlying classical motion by Blum *et al* (*Phys. Rev. Lett.*, **88** (2002), 114101) and by Bogomolny and Schmit (*Phys. Rev. Lett.*, **88** (2002), 114102). These statistics were shown to be computable from the random wave model of the eigenfunctions. We here study the analogous problem for chaotic maps whose phase space is the two-torus. We show that the distributions of the numbers of nodal points and nodal domains of the eigenvectors of the corresponding quantum maps can be computed straightforwardly and exactly using random matrix theory. We compare the predictions with the results of numerical computations involving quantum perturbed cat maps.

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In a recent article Blum *et al* (2002) observed that the number-distributions of the nodal domains of quantum wavefunctions of billiards whose classical dynamics is integrable are different from those for chaotic billiards and argued that the latter are universal. Thus, the number-distribution of nodal domains appears to be a new criterion for quantum chaos that complements the usual ones based on spectral fluctuations. Blum *et al* computed these distributions for some integrable (and separable) systems, but no analytic formula exists for the number of nodal domains of a chaotic billiard. Berry (1977) has conjectured that the wavefunctions of quantum systems with a chaotic classical limit behave like gaussian random functions. Supported by numerical evidence, Blum *et al* found that the limiting distribution of the number of nodal domains can be reproduced assuming Berry's conjecture. Bogomolny and Schmit (2002) developed a percolation model for nodal domains of gaussian random functions and showed that their number is gaussian distributed. They computed the mean and variance of this distribution, which are both proportional to the mean spectral counting function. Their

results agree with the numerical computations reported by Blum *et al* for chaotic billiards. The influence of a boundary on the nodal lines of gaussian random functions has been investigated by Berry (2002), Gnutzmann *et al* (2002), and Berry and Ishio (2002). This is expected to model the nodal properties of billiard wavefunctions near boundaries.

We here consider the analogous problem for one-dimensional time-reversal-symmetric systems with discrete time evolution and whose phase space is the two-dimensional torus \mathbb{T}^2 . The classical dynamics of such systems corresponds to the action of symplectic maps on \mathbb{T}^2 , and their quantum mechanics to that of unitary matrices U_N (called propagators or quantum maps) on a Hilbert space of dimension $N = 1/h$, where h is Planck's constant. Modelling the eigenvectors of U_N by those of random unitary symmetric matrices (such matrices constitute the circular orthogonal ensemble, COE, of random matrix theory), we compute the number-distributions of nodal domains and nodal points (the analogues of nodal lines in billiards) exactly. It is shown that these become gaussian as $N \rightarrow \infty$ and that the mean and variance are proportional to N (precisely as in the billiard case). We compare our results with numerical computations involving the eigenvectors of perturbations of quantum cat maps whose classical dynamics are hyperbolic and whose spectral statistics are known to be accurately predicted by random matrix theory (Basilio de Matos and Ozorio de Almeida, 1995; Keating and Mezzadri, 2000).

Consider the Helmholtz equation with Dirichlet boundary conditions

$$-\Delta\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \quad \mathbf{r} \in \Omega, \quad (1)$$

where Ω is a connected compact domain in a two-dimensional Riemann manifold. The *nodal lines* are the zero sets of real solutions of equation (1); the *nodal domains* are connected domains in Ω where $\Psi(\mathbf{r})$ has constant sign. Now, let $\{\Psi_n(\mathbf{r})\}_{n=1}^{\infty}$ be a set of eigenfunctions of the laplacian on Ω ordered by the magnitude of the corresponding eigenvalue E_n , and let ν_n be the number of nodal domains of the n -th eigenfunction. Courant (1923) proved that $\nu_n \leq n$. Let $I_g(E) = [E, E + gE]$, for $g > 0$. Blum *et al* (2002) introduced the distribution

$$P_b(x, I_g(E)) = \frac{1}{N_I} \sum_{E_n \in I_g(E)} \delta\left(x - \frac{\nu_n}{n}\right), \quad (2)$$

where N_I is the number of energy levels in $I_g(E)$. The limiting distribution of nodal domains is defined by

$$P_b(x) = \lim_{E \rightarrow \infty} P_b(x, I_g(E)). \quad (3)$$

We now introduce a density that is the analogue of (3) for quantum maps. The periodicity of the two-torus constrains the wavefunction to be an infinite sum of delta-functions supported at rational points of the form j/N , with j integer, in both the position and momentum basis (Hannay and Berry, 1980), i.e.

$$\psi(q) = \sum_{m \in \mathbb{Z}} \sum_{j=1}^N c_j \delta\left(q - \frac{j}{N} + m\right), \quad (4a)$$

$$\hat{\psi}(p) = \sum_{m \in \mathbb{Z}} \sum_{j=1}^N \hat{c}_j \delta \left(p - \frac{j}{N} + m \right), \quad (4b)$$

where $N = 1/h$ and

$$\hat{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(q) e^{-\frac{iqp}{\hbar}} dq. \quad (5)$$

Moreover, since $\psi(q)$ and $\hat{\psi}(p)$ are periodic, $c_j = c_{j+N}$ and $\hat{c}_j = \hat{c}_{j+N}$. Therefore, a quantum state is completely determined by N complex numbers, which implies that the Hilbert space is isomorphic to \mathbb{C}^N . The coefficient c_j can thus be interpreted as the value of $\psi(q)$ at $q = j/N$ (the Heisenberg uncertainty principle is not violated, because the periodic sum of delta-functions that defines $\hat{\psi}(p)$ extends to infinity). Now, let U_N be the matrix realization of a quantum map in the basis $\{|j\rangle\}_{j=1}^N$, where

$$\langle q | j \rangle = \sum_{m \in \mathbb{Z}} \delta \left(q - \frac{j}{N} + m \right). \quad (6)$$

We shall consider only systems whose dynamics is invariant under time reversal, so that U_N is a symmetric unitary matrix and, without loss of generality, the eigenvectors can be taken to be real.

Because of the topology of the phase space, an eigenvector of U_N is equivalent to a sequence of N real numbers with periodic boundary conditions, i.e. $c_1 = c_{N+1}$. A nodal point is then identified whenever two consecutive coefficients c_j have opposite sign. The total number of nodal points in a given eigenvector is

$$\nu = \frac{1}{2} \sum_{j=1}^N [1 - \text{sgn}(c_j) \text{sgn}(c_{j+1})], \quad (7)$$

where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \quad (8)$$

Similarly, a nodal domain is a set of consecutive integers $\{j+1, j+2, \dots, j+k\}$ such that the corresponding coefficients c_j lie between two nodal points and thus have constant sign. As a consequence of the periodicity of the coefficients c_j , there can be only an even number of nodal points, equal to the number of nodal domains; the only exception is when there are no nodal points and only one nodal domain. It follows from the results to be presented later that as $N \rightarrow \infty$ the probability that all the c_j s have the same sign is negligible, and so we shall denote by ν both the number of nodal points and the number of nodal domains. Finally, the limiting distribution is defined by

$$P_m(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \delta \left(x - \frac{\nu_n}{N} \right), \quad (9)$$

where, as for billiards, ν_n is the number of nodal domains (points) of the n -th eigenvector. Identical definitions can obviously be formulated in the momentum representation.

When the classical limit of U_N is a chaotic map, the eigenstatistics of U_N are expected to be the same as those of matrices in the COE (Bohigas *et al.*, 1984). The COE probability measure is invariant under the mapping

$$U \mapsto O U O^T, \quad (10)$$

where U is a unitary symmetric matrix and O is an arbitrary orthogonal matrix. Hence, each eigenvector of U is mapped by an orthogonal transformation into an eigenvector of a new matrix that by (10) has the same weight in the ensemble as U and the same spectrum. As a consequence (see, e.g. Haake, 2000), the eigenvectors of matrices in the COE are uniformly distributed on the unit sphere in \mathbb{R}^N and the joint probability density of their components is

$$P_{\text{COE}}(c_1, c_2, \dots, c_N) = \frac{1}{2\pi^{N/2}} \Gamma\left(\frac{N}{2}\right) \delta\left(1 - \sum_{j=1}^N c_j^2\right). \quad (11)$$

The above distribution is independent of the signs of the c_j s, therefore they can be either positive or negative with equal probability and there are no correlations among the signs of different coefficients. This simple observation allows us to compute analytically all the relevant quantities in a very straightforward way.

The signs of the c_j s behave like a sequence of N independent random variables s_j that assume the values $\{1, -1\}$ with equal probability $\frac{1}{2}$; in other words, they are equivalent to an array of non-interacting particles with spin $\frac{1}{2}$ and periodic boundary conditions. Thus, the probability of a configuration with N_+ spins up and $N_- = N - N_+$ spins down is given by the binomial distribution

$$P(N_+, N_-) = \frac{1}{2^N} \binom{N}{N_+}. \quad (12)$$

The computation of the density (9) requires a simple combinatorial argument. In a periodic chain of N spins there are N possible positions where a nodal point can be located. Hence, the number of configurations with ν nodal points is zero when ν is odd and twice the number of ways of choosing ν objects among N , irrespective of their ordering, for even ν , i.e.

$$[1 + (-1)^\nu] \binom{N}{\nu}. \quad (13)$$

The factor of two in front of the binomial coefficient is due to the fact that by simultaneously changing the sign of all the spins in the chain we obtain a new configuration with the nodal points in the same positions. Finally, the distribution of the number of nodal points and nodal domains is given by

$$P_m(\nu, N) = \frac{1 + (-1)^\nu}{2^N} \binom{N}{\nu}. \quad (14)$$

The mean $\langle \nu \rangle$ and variance $\sigma^2 = \langle \nu^2 \rangle - \langle \nu \rangle^2$ can be easily computed:

$$\langle \nu \rangle = \frac{N}{2} \quad \text{and} \quad \sigma^2 = \frac{N}{4}. \quad (15)$$

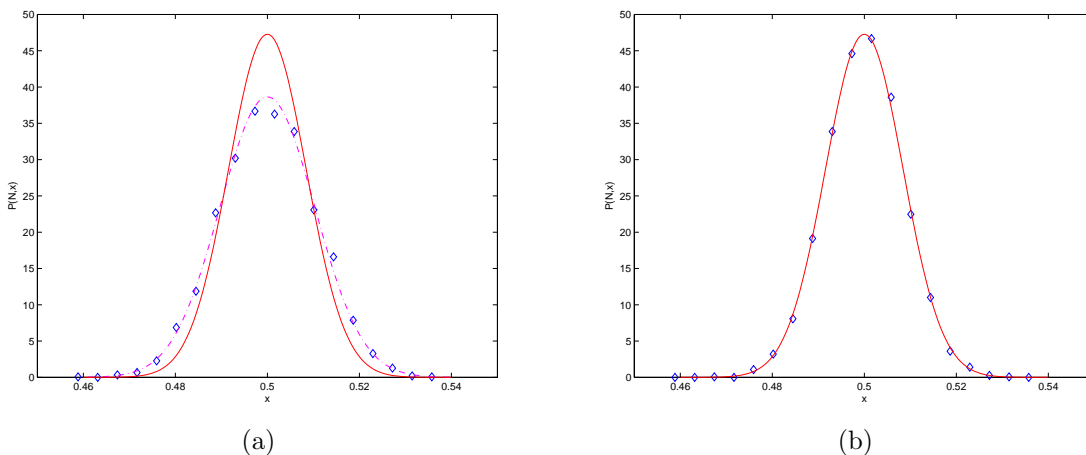


Figure 1. Nodal domain distributions (\diamond) of the unperturbed quantum cat map (a) and of its perturbation (20) with $k = 0.30$ (b) compared with the gaussian (16) (—). The dimension of the Hilbert space is $N = 3511$.

Equations (14) and (15) correspond to the results that Bogomolny and Schmit (2002) obtained for the percolation model of random wave functions in two-dimensional systems. By letting $N \rightarrow \infty$ and scaling $x = \nu/N$, the discrete distribution (14) tends to a continuous gaussian probability density with mean $\frac{1}{2}$ and variance $\sigma^2 = 1/4N$, i.e.

$$P_m(x, N) \sim \sqrt{\frac{2N}{\pi}} \exp[-2N(x - 1/2)^2], \quad N \rightarrow \infty. \quad (16)$$

This is the main result of this note.

In order to compare the distribution (16) with numerical computations, we consider perturbations of the following hyperbolic (cat) map:

$$A : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \pmod{1}. \quad (17)$$

Because of the number-theoretical properties of A , the spectrum of the propagator $U_N(A)$ is non-generic (Keating, 1991; Kurlberg and Rudnick, 2000) in that it does not obey the random matrix theory conjecture. However, if a small nonlinear perturbation is introduced, the composite map is still hyperbolic but loses its arithmetical nature. As a consequence, the spectrum of the new quantum map has random matrix correlations. Hence, we perturb (17) with the following shear in the momentum

$$\rho : \begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} q \\ p + \frac{k}{4\pi} \cos(2\pi q) \end{pmatrix} \quad (18)$$

and study the propagator $U_N(\phi)$ of the map

$$\phi = \rho \circ A \circ \rho. \quad (19)$$

The matrix elements of this propagator in the basis (6) are

$$U_N(\phi)_{lm} = \frac{1}{\sqrt{iN}} \exp \left\{ \frac{2\pi i}{N} \left[l^2 - lm + m^2 + \frac{N^2 k}{8\pi^2} (\sin(2\pi l/N) + \sin(2\pi m/N)) \right] \right\} \quad (20)$$

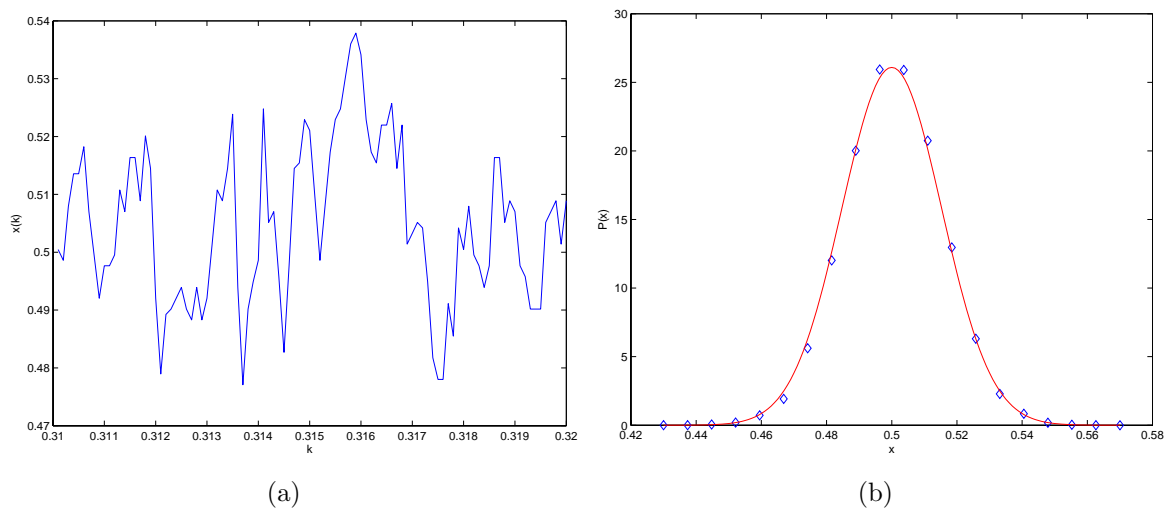


Figure 2. (a) Scaled number of nodal domains $x(k) = \nu(k)/N$ of an eigenvector of the matrix (20), with $N = 1069$, as a function of the perturbation parameter k ; (b) value distribution of $x(k)$ averaged over all eigenvectors (\diamond) compared with the gaussian (16) (—).

(Basilio de Matos and Ozorio de Almeida, 1995). It can be shown that the only symmetry of this quantum map is time reversal (Keating and Mezzadri, 2000). Furthermore, if $k < k_{\max} = 0.32\dots$, then the map (19) is uniformly hyperbolic and the spectral statistics of the propagator (20) are consistent with random matrix theory (Basilio de Matos and Ozorio de Almeida, 1995). Figure 1(b) shows the nodal domain distribution of the eigenvectors of the quantum map (20) for a particular choice of k and N , together with the density (16). The nodal domain distribution of the unperturbed quantum map, figure 1(a), also appears to be gaussian, but its variance cannot be predicted by random matrix theory.

As the perturbation parameter k varies, the nodal points in a given eigenvector of the matrix (20) change their positions. A natural question then arises: what is the minimum number of parameters needed to create or coalesce nodal points and alter the number of nodal domains? In other words, what is the codimension of the nodal points? Since the spins in a chain are uncorrelated, the functions $s_j(k)$ will be independent, therefore nodal points move randomly without repelling or attracting each other. Thus, the codimension of nodal points is one and a single parameter is enough to create or annihilate nodal domains with equal probability. This behaviour is illustrated in figure 2; the scaled number of nodal domains $x(k) = \nu(k)/N$ of an eigenvector oscillates around $\frac{1}{2}$, and since the $s_j(k)$ are independent, the value distribution of $x(k)$ is given by the gaussian (16).

Finally, it is worth remarking that this problem is equivalent to a one-dimensional Ising model of *non-interacting* spins in a magnetic field B with periodic boundary

conditions, whose Hamiltonian and partition function are

$$H = -B \sum_{j=1}^N s_j, \quad s_j = \pm 1, \quad s_1 = s_{N+1} \quad (21)$$

and

$$Z(\beta, B) = \sum_{\{s_1\}} \sum_{\{s_2\}} \dots \sum_{\{s_N\}} \exp(-\beta H) = 2^N \cosh(\beta B)^N \quad (22)$$

respectively. All the relevant thermodynamical quantities should be computed at $\beta = B = 0$. This plays the role in this case of the analogy between the nodal statistics of billiard wavefunctions and the Potts model suggested by Bogomolny and Schmit (2002).

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