On an average over the Gaussian Unitary Ensemble *

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Abstract

We study the asymptotic limit for large matrix dimension N of the partition function of the unitary ensemble ($\beta = 2$) with weight

$$w(x) := \exp\left(-\frac{z^2}{2x^2} + \frac{t}{x} - \frac{x^2}{2}\right).$$

We compute the leading order term of the partition function and of the coefficients of its Taylor expansion. Our results are valid in the region $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$. Such partition function contains all the information on a new statistics of the eigenvalues of matrices in the Gaussian Unitary Ensemble (GUE) that was introduced by Berry and Shukla [2]. It can also be interpreted as the moment generating function of the singular linear statistics

$$\sum_{j=1}^{N}\left(\frac{1}{x_j} + \frac{1}{x_j^2}\right).$$

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1 Introduction

1.1 Background

In Random Matrix Theory partition functions of ensembles whose probability measure is invariant under conjugation by unitary matrices (unitary or $\beta = 2$ ensembles) are integrals of the form

$$\frac{1}{N!}\int_{J^N}\prod_{j=1}^{N} w(x_j)\prod_{1\le j<k\le N}\left|x_k - x_j\right|^2 d^N x, \tag{1.1}$$

where $w(x) \geq 0$ is a weight function and usually J is either an interval, or the whole real line or the unit circle. The theory of orthogonal polynomials (see Szegő [15], pp. 23–28) implies that such integrals are determinants of Hankel or Toeplitz matrices, or a

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linear combination of the two. Computing asymptotic formulae of such determinants is a very important task — often a very difficult one — in many branches of mathematics and physics. The asymptotics of the integral (1.1) depends crucially on the analytic properties of $w(x)$. Usually, singular weights are the most challenging.

The purpose of this article is to compute the following expectation value over the Gaussian Unitary Ensemble (GUE):

$$E_N(z,t) := \int_{\mathbb{R}^N} \left(\prod_{j=1}^{N} \exp\left(-\frac{z^2}{2x_j^2} + \frac{t}{x_j} \right) \right) P_{\mathrm{GUE}}(x_1,\ldots,x_N) d^N x, \tag{1.2}$$

where

$$P_{\mathrm{GUE}}(x_1,\ldots,x_N) := \frac{1}{Z_N N!} \exp\left(-\frac{1}{2}\sum_{j=1}^{N} x_j^2 \right) \prod_{1\le j<k\le N} |x_k - x_j|^2 \tag{1.3}$$

is the joint probability density function (*j.p.d.f.*) of the eigenvalues and

$$\begin{aligned} Z_N &:= \frac{1}{N!}\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\sum_{j=1}^{N} x_j^2 \right) \prod_{1\le j<k\le N} |x_k - x_j|^2 \, d^N x \\ &= (2\pi)^{N/2} \prod_{j=1}^{N-1} j! \end{aligned} \tag{1.4}$$

is the partition function of the GUE.

Let

$$\Lambda_N(x) := \prod_{j=1}^{N} (x - x_j), \tag{1.5}$$

be a polynomial of degree N whose roots $x_1,\ldots,x_N$ are all real and define

$$Q_N(x) := \frac{\Lambda_N'^2(x)}{\Lambda_N'^2(x) - \Lambda_N(x)\Lambda_N''(x)}. \tag{1.6}$$

This function was studied by Tuck [16] in a numerical investigation of the zeros of the Riemann zeta function $\zeta(s)$. In Tuck's article $\Lambda_N(x)$ was replaced by the Hardy function:

$$Z(t) := t^{1/4} \exp\left(\tfrac{\pi t}{4}\right) \pi^{-\frac{1}{2}it} \Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}it\right) \zeta\left(\tfrac{1}{2} + it\right). \tag{1.7}$$

The Hardy function is an entire function of order one; it is real for $t \in \mathbb{R}$ and its zeros coincide with non trivial zeros of $\zeta(s)$. The motivation to study (1.6) was that the Riemann hypothesis implies that

$$W(t) := Z'^2(t) - Z(t)Z''(t) > 0. \tag{1.8}$$

More generally, if $Z(t)$ is an entire function of order less or equal to one, real on $\mathbb{R}$ and whose zeros are all real then $W(t) > 0$.

Suppose that the roots of $\Lambda_N(x)$ are N random variables with *j.p.d.f.* $p(x_1,\dots,x_N)$, and denote by $P(Q_N)$ the probability density function (*p.d.f.*) of the random variable $Q_N(x)$. Berry and Shukla [2] observed that $Q_N(x)$ is a sensitive indicator of the degree of repulsion between neighbouring zeros of $\Lambda_N(x)$. More precisely, the rate of decay of $P(Q_N)$ as $Q_N \to \infty$ is related to the rigidity of the roots of $\Lambda_N(x)$. They also studied in detail $P(Q_N)$ in two cases: when the roots of $\Lambda_N(x)$ are N independent identically distributed (*i.i.d.*) standard normal random variables and when $\Lambda_N(x)$ is the characteristic polynomial of a matrix in the GUE.

Berry and Shukla showed that all the information on $P(Q_N)$ is contained in the expectation value

$$\int_{\mathbb{R}^N}\left(\prod_{j=1}^{N}\exp\left(-\frac{z^2}{2x_j^2}+\frac{t}{x_j}\right)\right)p(x_1,\dots,x_N)d^Nx, \tag{1.9}$$

in the sense that all the moments of the distribution $P(Q_N)$ can be extracted from its knowledge. They computed the average (1.9) and the moments of $P(Q_N)$ when the roots of $\Lambda_N(x)$ are *i.i.d.* standard normal random variables, but not when they are the eigenvalues of matrix in the GUE.

The integral in the right-hand side of equation (1.2) is amenable to other interpretations. For example, if we set

$$s = z^2/2 = -t, \quad s > 0, \tag{1.10}$$

then

$$M_N(s) := \int_{\mathbb{R}^N}\prod_{j=1}^{N}\exp\left(-s\left(\frac{1}{x_j^2}+\frac{1}{x_j}\right)\right)P_{\mathrm{GUE}}(x_1,\dots,x_N)d^Nx \tag{1.11}$$

is the moment generating function of the *p.d.f.* of the singular linear statistics

$$\sum_{j=1}^{N}\left(\frac{1}{x_j^2}+\frac{1}{x_j}\right). \tag{1.12}$$

Furthermore, $Z_N E_N(z,t)$ is the partition function of the unitary ensemble with weight

$$w(x) := \exp\left(-\frac{z^2}{2x^2}+\frac{t}{x}-\frac{x^2}{2}\right). \tag{1.13}$$

Now, denote by $\pi_j(x)$, $j \in \mathbb{Z}_+$, the monic polynomials orthogonal with respect to $w(x)$. The integral $E_N(z,t)$ can be rewritten as a Hankel determinant:

$$E_N(z,t) = Z_N^{-1}\det\left(\mu_{j+k}\right)_{j,k=0}^{N-1} = Z_N^{-1}\prod_{j=0}^{N-1}h_j, \tag{1.14}$$

where

$$\mu_j := \int_{-\infty}^{\infty} w(x)x^j dx, \quad j \in \mathbb{Z}_+ \tag{1.15}$$

and

$$\int_{-\infty}^{\infty} w(x)\pi_j(x)\pi_k(x)dx = h_j\delta_{jk}. \quad (1.16)$$

If we know the behaviour of the polynomials $\pi_N(x)$ as $N \to \infty$ (Plancherel-Rotach asymptotics), then we can extract information on the asymptotic limit of $E_N(z,t)$. Our approach consists in studying the solution of the Riemann-Hilbert (R-H) problem associated to the polynomials $\pi_N(x)$ in the limit as $N \to \infty$. The main tool is the nonlinear steepest descent analysis developed by Deift *et al* [8, 9]. The average $E_N(z,t)$ can then be computed in terms of such a solution using a set of differential identities introduced by Bertola *et al* [3].

After this work was completed, we discovered that independently Chen and Its [5, 6] studied the orthogonal polynomial problem associated with the partition function

$$H_N(\alpha, s) := \frac{1}{N!}\int_{[0,\infty)^N} \prod_{j=1}^{N} x_j^{\alpha} e^{-x_j - s/x_j} \prod_{1\le j<k\le N} |x_k - x_j|^2 \, d^N x, \quad (1.17)$$

where $\alpha > -1$ and $s \ge 0$. The weight in this integral is that one of the Laguerre polynomials perturbed by the singular factor $e^{-s/x}$. In a first paper Chen and Its [5] proved that $H_N(\alpha, s)$ can be expressed as the integral of the combination of particular third Painlevé functions; in a second article [6] they derived asymptotic formulae for the orthogonal polynomials, the corresponding recurrence coefficients and the h_j's. When the parameters t and α in equation (1.13) and $x^{\alpha}e^{-x-s/x}$ respectively are both set equal to zero, then the systems of orthogonal polynomials associated with the two weights can be mapped into each other by the change of variables $x \mapsto x^2/2$. The Plancherel-Rotach asymptotics of the orthogonal polynomials can be studied using the nonlinear steepest descent in both cases. However, even when $t = \alpha = 0$, the partition functions (1.2) and (1.17) are not equivalent and cannot be mapped into each other by a simple change of variables.

1.2 Statement of results

The average (1.2) is an entire function of t, thus its Taylor series has an infinite radius of convergence and we can write

$$E_N(z,t) = \sum_{m=0}^{\infty} E_{Nm}(z)t^m. \quad (1.18)$$

The main goal of this paper is to compute the leading order term in the asymptotic expansion of $E_N(z,t)$ and $E_{Nm}(z)$ as $N \to \infty$. Our main result is the following:

Theorem 1.1. *Let $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$, where c_1 and c_2 are two constants independent of z and N. The expectation value $E_N(z,t)$ is given by*

$$\begin{aligned} E_N(z,t) = B_N \exp\left(\frac{z^2}{4} - \frac{9}{2^{\frac{10}{3}}}\left(N^{\frac{2}{3}} z^{\frac{4}{3}} - 1\right) + \frac{t^2 N^{\frac{1}{3}}}{2^{\frac{5}{3}} z^{\frac{4}{3}}}\right) \\ \times \left(1 + o(1)\right), \quad N \to \infty, \end{aligned} \quad (1.19)$$

where B_N is the ensemble average

$$B_N := \int_{\mathbb{R}^N} \left(\prod_{j=1}^{N} e^{-\frac{1}{2Nx_j^2}} \right) P_{\mathrm{GUE}}(x_1, \ldots, x_N) d^N x, \tag{1.20}$$

which is independent of t.

Berry and Shukla [2] showed that the m-th moment of $P(Q_N)$ is given by the integral

$$M_{Nm} := 2^{1-m} \prod_{n=m}^{2m} n \int_0^\infty z^{2m-1} E_{N2m}(z) dz. \tag{1.21}$$

From Theorem 1.1 it is straightforward to compute the coefficient $E_{N2m}(z)$ in the series expansion (1.18).

Corollary 1.1. *Let $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$, where c_1 and c_2 are two constants independent of z and N. The leading order term of the coefficient of t^{2m} in equation* (1.18) *is*

$$E_{N2m}(z) \sim B_N \exp\left(\frac{z^2}{4} - \frac{9}{2^{\frac{10}{3}}} \left(N^{\frac{2}{3}} z^{\frac{4}{3}} - 1 \right) \right) \frac{N^{\frac{m}{3}}}{2^{\frac{5m}{3}} m! z^{\frac{4m}{3}}}, \quad N \to \infty. \tag{1.22}$$

Unfortunately, the asymptotic limit in equations (1.19) and (1.22) cannot be assumed to be uniform in z: there may be non negligible contributions from the region $z < c_1 N^{-\frac{1}{2}}$ that would affect the integral (1.21). Such contributions can be investigated by studying the double scaling limit of the matrix ensemble with weight (1.13). This will be the subject of a forthcoming publication.

The structure of this article is the following: in §2 we introduce the R-H problem for the orthogonal polynomials with weight (1.13) (after appropriate rescaling of x, z and t) and the differential identities used to compute the leading order asymptotics of $E_N(z,t)$; in §3 we find the equilibrium measure on which the R-H analysis is based; in §4 we apply the nonlinear steepest descent to the R-H problem; §5 and §6 are devoted to complete the proof of Theorem 1.1 combining the asymptotics of the solution of the R-H problem and the differential identities discussed in §2.

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2 Preliminaries

Let us introduce the scaling

$$v_1 = \frac{t}{\sqrt{N}}, \quad v_2 = \left(\frac{z}{N}\right)^2 \quad \text{and} \quad y_j = \frac{x_j}{\sqrt{N}}, \quad j = 1, \dots N. \tag{2.1}$$

The weight (1.13) becomes

$$w_N(y) := \exp\left(-N\left(\frac{v_2}{2y_j^2} + \frac{y_j^2}{2}\right) + \frac{v_1}{y_j}\right). \tag{2.2}$$

We also define the partition function

$$G_N(v_1, v_2) := \frac{1}{N!}\int_{\mathbb{R}^N} \prod_{j=1}^{N} \exp\left(-N\left(\frac{v_2}{2y_j^2} + \frac{y_j^2}{2}\right) + \frac{v_1}{y_j}\right) \prod_{1\le j<k\le N}^{N} |y_k - y_j|^2 \, d^N y, \tag{2.3}$$

which is proportional to the average $E_N(z,t)$, namely

$$E_N(z,t) = Z_N^{-1} N^{\frac{N^2}{2}} G_N(v_1, v_2). \tag{2.4}$$

For convenience, where there is no risk of confusion with the quantities introduced in §1.1, we denote the polynomials orthogonal with respect to $w_N(y)$ by $\pi_j(y)$. Similarly, we write

$$G_N(v_1, v_2) = \det\left(\mu_{j+k}\right)_{j,k=0}^{N-1} = \prod_{j=0}^{N-1} h_j, \tag{2.5}$$

where

$$\mu_j := \int_{-\infty}^{\infty} w_N(y) y^j dx, \quad j \in \mathbb{Z}_+ \tag{2.6}$$

and

$$\int_{-\infty}^{\infty} w_N(y)\pi_j(y)\pi_k(y) dy = h_j \delta_{jk}. \tag{2.7}$$

Let us define the matrix valued function

$$Y(y) := \begin{pmatrix} \pi_N(y) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\pi_N(s) w_N(s)}{s-y} ds \\ \kappa_{N-1}\pi_{N-1}(y) & \frac{\kappa_{N-1}}{2\pi i}\int_{-\infty}^{\infty} \frac{\pi_{N-1}(s) w_N(s)}{s-y} ds \end{pmatrix}, \tag{2.8}$$

where $\kappa_{N-1} = -2\pi i h_{N-1}$. Fokas *et al* [10, 11] showed that $Y(y)$ solves the following R-H problem:

$$\begin{aligned} &1. \quad Y(y) \text{ is analytic in } \mathbb{C}/\mathbb{R}, \\ &2. \quad Y_+(y) = Y_-(y)\begin{pmatrix} 1 & w_N(y) \\ 0 & 1 \end{pmatrix}, \qquad y \in \mathbb{R}, \\ &3. \quad Y(y) = \left(I + O(y^{-1})\right)\begin{pmatrix} y^N & 0 \\ 0 & y^{-N} \end{pmatrix}, \qquad y \to \infty \end{aligned} \tag{2.9}$$

where $Y_+(y)$ and $Y_-(y)$ denotes the limiting values of $Y(y)$ as it approaches the left and right-hand side of the real axis. It turns out that the partition function $G_N(v_1, v_2)$ can be expressed in terms of $Y(y)$.

Lemma 2.1 (Bertola, Eynard and Hanard [3]). *The following differential identities hold:*

$$\frac{\partial \log G_N}{\partial v_1} = -\frac{1}{4\pi i}\oint_{y=0} \frac{1}{y}\,\mathrm{Tr}\left(Y^{-1}(y)Y'(y)\sigma_3\right) dy, \tag{2.10a}$$

$$\frac{\partial \log G_N}{\partial v_2} = \frac{N}{8\pi i}\oint_{y=0} \frac{1}{y^2}\,\mathrm{Tr}\left(Y^{-1}(y)Y'(y)\sigma_3\right) dy, \tag{2.10b}$$

where the contour of integration is a small loop around $y = 0$ oriented counter-clockwise.

Proof. Taking the logaritmic derivatives of both sides of equation (2.5) and using the orthogonality conditions (2.7) gives

$$\frac{\partial \log G_N}{\partial v_k} = \int_{-\infty}^{\infty} \left(\sum_{j=0}^{N-1} \frac{\pi_j^2(y)}{h_j}\right) \frac{\partial w_N(y)}{\partial v_k} dy, \quad k = 1, 2. \tag{2.11}$$

These integrals can be rewritten as

$$\frac{\partial \log G_N}{\partial v_k} = (-N)^{k-1} \int_{-\infty}^{\infty} \frac{K_N(y,y)}{k y^k} dy, \quad k = 1, 2, \tag{2.12}$$

where $K_N(x, y)$ is the kernel

$$K_N(x,y) := \sqrt{w_N(x)w_N(y)} \sum_{j=0}^{N-1} \frac{\pi_j(x)\pi_j(y)}{h_j} \tag{2.13}$$

and

$$G_N(v_1, v_2) = \left(N! \prod_{j=0}^{N-1} h_j\right) \det\left(K_N(y_j, y_k)\right)_{j,k=0}^{N-1}. \tag{2.14}$$

In order to evaluate the integral (2.12) we use the relation

$$\begin{aligned} K_N(x,y) &= \frac{\sqrt{w_N(x)w_N(y)}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\sqrt{w_N(x)w_N(y)}}{2\pi i(x-y)} \mathrm{Tr}\left(Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right), \end{aligned} \tag{2.15}$$

which follows from the Christoffel-Darboux formula, the definition of $Y(x)$ and the fact that $\det Y(y) = 1$. By using l'Hospital's rule we obtain

$$K_N(y,y) = \frac{w_N(y)}{2\pi i} \mathrm{Tr}\left(Y_+^{-1}(y) Y_+'(y) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right). \tag{2.16}$$

Combining the jump condition of the R-H problem (2.9) and equation (2.16) gives

$$K_N(y,y) = \frac{1}{4\pi i}\Big(\mathrm{Tr}\left(Y_-^{-1}(y)Y_-'(y)\sigma_3\right) - \mathrm{Tr}\left(Y_+^{-1}(y)Y_+'(y)\sigma_3\right)\Big), \quad y \in \mathbb{R}. \tag{2.17}$$

The asymptotic behaviour of $Y(y)$ as $y \to \infty$ implies that the functions

$$\mathrm{Tr}\left(Y_\pm^{-1}(y)Y_\pm'(y)\sigma_3\right) \tag{2.18}$$

are analytic in the upper/lower half planes with a simple pole at infinity. Finally, the identities (2.10) follow from the residue theorem. □

This lemma gives an explicit link between the solution of the R-H problem and the average (1.2). The main challenge that we are facing is to compute an asymptotic formula for $Y(y)$ using the nonlinear steepest descent method. Then, we can obtain a formula for $E_N(z,t)$ using equations (2.10).

3 The equilibrium measure

The asymptotics analysis of $Y(y)$ relies on computing the g-function and the support of the equilibrium measure for the potential

$$V_0(y) := \frac{v_2}{2y^2} + \frac{y^2}{2}, \tag{3.1}$$

where we have neglected the term v_1/y because it is asymptotically small. The equilibrium measure is the positive normalized Borel measure $\mu(y)$ that minimizes the energy function

$$I(\mu) := -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log|x-y|d\mu(x)d\mu(y) + \int_{-\infty}^{\infty} V_0(y)d\mu(y). \tag{3.2}$$

It satisfies the conditions

$$2\int_{-\infty}^{\infty} \log|y-s|d\mu(s) - V_0(y) = l, \qquad x \in \mathrm{Supp}\,(\mu)\,, \tag{3.3a}$$

$$2\int_{-\infty}^{\infty} \log|y-s|d\mu(s) - V_0(y) \le l, \qquad x \in \mathbb{R}/\,\mathrm{Supp}\,(\mu)\,. \tag{3.3b}$$

for some constant l. Moreover, if $\mu(y)$ satisfies (3.3), then it must be equal to the equilibrium measure.

The g-function is defined by

$$g(y) := \int_{-\infty}^{\infty} \log(y-s)d\mu(s). \tag{3.4}$$

It is analytic in $\mathbb{C}\setminus\mathbb{R}$ and has jump discontinuities on the real axis. The conditions (3.3) expressed in terms of $g(y)$ become

$$g_+(y)+g_-(y)-V_0(y)=l, \qquad y\in \mathrm{Supp}\,(\mu)\,, \tag{3.5a}$$

$$\mathrm{Re}\big(g_+(y)+g_-(y)\big)-V_0(y)\le l, \qquad y\in \mathbb{R}/\,\mathrm{Supp}\,(\mu)\,. \tag{3.5b}$$

Furthermore, we have

$$g(y)=\log y+O(y^{-1}), \quad y\to\infty. \tag{3.6}$$

We shall derive an expression for $g(y)$ using the ansatz

$$g'(y)=\frac{V_0'(y)}{2}-\frac{(y^2-\lambda_1^2)\sqrt{(y^2-\lambda_2^2)(y^2-\lambda_3^2)}}{2y^3}. \tag{3.7}$$

The constants λ_1, λ_2 and λ_3 are determined by the asymptotic behaviour of $g'(y)$, namely

$$g'(y)=\frac{1}{y}+O(y^{-2}), \qquad y\to\infty, \tag{3.8a}$$

$$g'(y)=O(1), \qquad y\to 0. \tag{3.8b}$$

These constraints give

$$\lambda_1^2+\frac{1}{2}\left(\lambda_2^2+\lambda_3^2\right)=2, \tag{3.9a}$$

$$\lambda_1^{-2}+\frac{1}{2}\left(\lambda_2^{-2}+\lambda_3^{-2}\right)=0, \tag{3.9b}$$

$$\lambda_1^2\lambda_2\lambda_3=-v_2. \tag{3.9c}$$

Let $A_j=\lambda_j^2$. Then, we see that A_2, A_3 are the solutions of the quadratic equation

$$y^2-2(2-A_1)y+\frac{v_2^2}{A_1^2}=0, \tag{3.10}$$

while A_2^{-1} and A_3^{-1} are solutions of

$$y^2+\frac{2}{A_1}y+\frac{A_1^2}{v_2^2}=0. \tag{3.11}$$

From equation (3.10) it follows that A_2^{-1} and A_3^{-1} also satisfy

$$\frac{v_2^2}{A_1^2}y^2-2(2-A_1)y+1=0. \tag{3.12}$$

By comparing this equation with (3.11), we see that A_1 is a solution of the equation

$$A_1^4-2A_1^3-v_2^2=0. \tag{3.13}$$

Lemma 3.1. *There exists a solution to equation* (3.9) *such that* $\lambda_1 \in i\mathbb{R}$ *and* $\lambda_2, \lambda_3 \in \mathbb{R}$.

Proof. The solutions of (3.11) are

$$y = -\frac{1}{A_1}\left(1 \pm \sqrt{1 - \left(\frac{A_1}{\sqrt{v_2}}\right)^4}\right) \tag{3.14}$$

Therefore, if a solution of (3.13) such that $-\sqrt{v_2} < A_1 < 0$ exists, then the two solutions to (3.14) are real and positive and the lemma is proven. Now, if $A_1 = 0$, then the left-hand side of (3.13) is $-v_2^2 < 0$; if $A_1 = -\sqrt{v_2}$ it is $2v_2^{\frac{3}{2}} > 0$. Hence, there is a solution to (3.13) between $-\sqrt{v_2}$ and 0. □

Let us now choose λ_1, λ_2 and λ_3 such that $\mathrm{Im}(\lambda_1) > 0$ and $0 < \lambda_2 < \lambda_3$. Furthermore, define

$$g(y) := \frac{V_0(y)}{2} - \int_{\lambda_3}^{y} \frac{(s^2 - \lambda_1^2)\sqrt{(s^2 - \lambda_2^2)(s^2 - \lambda_3^2)}}{2s^3} ds + \frac{l}{2}, \tag{3.15}$$

where the integration path is chosen such that it does not intersect the interval $(-\infty, \lambda_3)$ and

$$l = -2 \lim_{y\to\infty}\left(\frac{V_0(y)}{2} - \log y - \int_{\lambda_3}^{y} \frac{(s^2 - \lambda_1^2)\sqrt{(s^2 - \lambda_2^2)(s^2 - \lambda_3^2)}}{2s^3} ds\right). \tag{3.16}$$

The function $g(y)$ is analytic in $\mathbb{C} \setminus (-\infty, \lambda_3)$ and $g(y) \sim \log y$ as $y \to \infty$. Now, let $\Sigma := \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 := [-\lambda_3, -\lambda_2] \quad \text{and} \quad \Sigma_2 := [\lambda_2, \lambda_3]\,. \tag{3.17}$$

We will dedicate the rest of this section to show that $g(y)$ satisfies the conditions (3.5) with

$$\mathrm{Supp}\,(\mu) = \Sigma = \Sigma_1 \cup \Sigma_2. \tag{3.18}$$

Lemma 3.2. *The function* $g(y)$ *defined in equation* (3.15) *satisfies the jump discontinuities*

$$g_+(y) + g_-(y) - V_0(y) = l, \quad y \in \Sigma, \tag{3.19a}$$

$$g_+(y) - g_-(y) = 2\pi i, \quad y \in (-\infty, -\lambda_3), \tag{3.19b}$$

$$g_+(y) - g_-(y) = \pi i, \quad y \in (-\lambda_2, \lambda_2). \tag{3.19c}$$

Proof. Since $V_0(y)/2$ has no jump discontinuities in $\mathbb{C}$, we only concentrate on the integral in the right-hand side of (3.15). Let us write

$$\nu(y) := \frac{(y^2 - \lambda_1^2)\sqrt{(y^2 - \lambda_2^2)(y^2 - \lambda_3^2)}}{2y^3}, \tag{3.20a}$$

$$\tilde{g}(y) := \int_{\lambda_3}^{y} \nu(s) ds. \tag{3.20b}$$

If $y \in \Sigma_2$, then

$$\tilde{g}_+(y) + \tilde{g}_-(y) = \int_{\lambda_3}^{y} \big(\nu_+(s) + \nu_-(s)\big) ds. \tag{3.21}$$

Since $\nu(y)$ changes sign across Σ, we have

$$\nu_+(y) = -\nu_-(y), \quad y \in \Sigma. \tag{3.22}$$

Equations (3.21) and (3.22) imply that (3.19a) is satisfied for $y \in \Sigma_2$.

Suppose now that $y \in \Sigma_1$. Let $\Gamma_\pm$ be a contour that consists of 3 parts: the first one $\Gamma^1_\pm$ goes from λ_3 to λ_2 on the positive/negative side of the real axis; the second part $\Gamma^2_\pm$ is a semicircle from λ_2 to $-\lambda_2$ in the upper/lower half plane; the last part $\Gamma^3_\pm$ goes from $-\lambda_2$ to y on the positive/negative side of the real axis. Then, the jump on Σ_1 is given by

$$\begin{aligned} \tilde{g}_+(y) + \tilde{g}_-(y) = \int_{\Gamma_+} \nu(s) ds + \int_{\Gamma_-} \nu(s) ds = \int_{-\lambda_2}^{y} \big(\nu_+(s) + \nu_-(s)\big) ds \\ + \int_{\lambda_3}^{\lambda_2} \big(\nu_+(s) + \nu_-(s)\big) ds + \int_{\Gamma^2_+} \nu(s) ds + \int_{\Gamma^2_-} \nu(s) ds. \end{aligned} \tag{3.23}$$

Since $\nu_+(s) = -\nu_-(s)$ on Σ, the first two terms in the right-hand side of equation (3.23) are zero. Furthermore, under the map $s \mapsto -s$, the contour Γ^2_+ becomes $-\Gamma^2_-$ and $\nu(-s) = -\nu(s)$. Hence, the sum of the last two terms are zero too. This proves (3.19a) on Σ_1.

Let $y \in (-\infty, -\lambda_3)$. We simply have

$$\tilde{g}_+(y) - \tilde{g}_-(y) = 2\pi i \operatorname*{Res}_{y=\infty} \tilde{g}(y). \tag{3.24}$$

Then, equation (3.19b) follows from

$$\operatorname*{Res}_{y=\infty} \tilde{g}(y) = -\frac{\lambda_1^2 + (\lambda_2^2 + \lambda_3^2)/2}{2} = -1, \tag{3.25}$$

where we have used (3.9a).

Let $y \in (-\lambda_2, \lambda_2)$ and consider the contours $\tilde{\Gamma}_\pm := \Gamma^1_\pm \cup \tilde{\Gamma}^2_\pm$ and $\tilde{\Gamma}^3_\pm$. Now $\tilde{\Gamma}^2_\pm$ joins λ_2 to y and and $\tilde{\Gamma}^3_\pm$ connects $-\lambda_2$ to $-\lambda_3$ respectively. As before $\pm$ denotes the upper/lower half plane respectively. We have

$$\tilde{g}_+(y) - \tilde{g}_-(y) = \int_{\tilde{\Gamma}_+} \nu(s) ds - \int_{\tilde{\Gamma}_-} \nu(s) ds = \int_{\Gamma^1_+ \cup (-\Gamma^1_-)} \nu(s) ds, \tag{3.26}$$

where we have used

$$\operatorname*{Res}_{y=0} \tilde{g}(y) = \frac{\lambda_1^{-2} + \left(\lambda_2^{-2} + \lambda_3^{-2}\right)/2}{2} = 0, \tag{3.27}$$

which follows from equation (3.9b). The function $\nu(s)$ is odd and $\Gamma^1_\pm$ is mapped into $-\tilde{\Gamma}^3_\mp$ by $s \mapsto -s$, therefore

$$\int_{\tilde{\Gamma}^3_+ \cup (-\tilde{\Gamma}^3_-)} \nu(s) ds = \int_{\Gamma^1_+ \cup (-\Gamma^1_-)} \nu(s) ds. \tag{3.28}$$

Finally, Cauchy's theorem gives

$$\int_{\Gamma^1_+ \cup (-\Gamma^1_-)} \nu(s) ds = \pi i \operatorname*{Res}_{y=\infty} \tilde{g}(y) = -i\pi. \tag{3.29}$$

This completes the proof of equation (3.19c). □

We are now ready to prove that the definition (3.15) gives the g-function.

Proposition 3.1. *Suppose that λ_1, λ_2 and λ_3 satisfy the conditions of Lemma* 3.1, *that* $\mathrm{Im}\,(\lambda_1) > 0$ *and* $0 < \lambda_2 < \lambda_3$. *Then, the function $g(y)$ defined in* (3.15) *satisfies*

$$g_+(y) + g_-(y) - V_0(y) = l, \qquad y \in \Sigma, \tag{3.30a}$$

$$\mathrm{Re}\,(g_+(y) + g_-(y)) - V_0(y) < l, \qquad y \in \mathbb{R} \setminus \Sigma. \tag{3.30b}$$

Proof. Equation (3.30a) was proven in Lemma 3.2. We are left to prove inequality (3.30b). From the jump discontinuities (3.19), we see that outside Σ the real parts of $g_+(y)$ and $g_-(y)$ are equal. In particular, the real part of $g(y)$ is continuous outside Σ.

Equation (3.19a) and

$$\tilde{g}(y) = \frac{V_0(y)}{2} - g(y) + \frac{l}{2} \tag{3.31}$$

give the equality

$$\mathrm{Re}\big(\tilde{g}_+(y) + \tilde{g}_-(y)\big) = 0, \quad y \in \Sigma. \tag{3.32}$$

Now note that

$$\begin{cases} \nu(y) > 0 & \text{if } y \in (-\lambda_2, 0) \cup (\lambda_3, \infty), \\ \nu(y) < 0 & \text{if } y \in (-\infty, -\lambda_3) \cup (0, \lambda_2). \end{cases} \tag{3.33}$$

Thus, $\mathrm{Re}\,\tilde{g}(y)$ is an increasing function in $(-\lambda_2, 0) \cup (\lambda_3, \infty)$ and decreases in $(-\infty, -\lambda_3) \cup (0, \lambda_2)$. This property and (3.32) imply

$$\mathrm{Re}\big(\tilde{g}_+(y) + \tilde{g}_-(y)\big) > 0, \quad x \in \mathbb{R} \setminus \Sigma. \tag{3.34}$$

The proposition now follows from equations (3.31) and (3.34). □

4 The Riemann-Hilbert analysis

The purpose of this section is to study the asymptotic limit of the solution $Y(y)$ of the R-H problem (2.9) using the nonlinear steepest descent analysis developed by Deift *et al* [8, 9]. One of the main ingredients is the g-function computed in §3.

4.1 Deformation of the Riemann-Hilbert problem

Let us introduce

$$F(y) := \frac{v_1 q(y)}{2\pi i}\left(\int_\Sigma \frac{ds}{sq_+(s)(s-y)} + \xi\int_{-\lambda_2}^{\lambda_2}\frac{ds}{q(s)(s-y)}\right), \tag{4.1}$$

where $q(y)$ and ξ are defined by

$$q(y) := \sqrt{(y^2-\lambda_2^2)(y^2-\lambda_3^2)}, \tag{4.2a}$$

$$K_0 := 2\int_{\lambda_2}^{-\lambda_2}\frac{ds}{q(s)}, \tag{4.2b}$$

$$\xi := -\frac{\int_\Sigma \frac{ds}{sq_+(s)}}{\int_{-\lambda_2}^{\lambda_2}\frac{ds}{q(s)}} = -\frac{2\pi i}{K_0\lambda_2\lambda_3}. \tag{4.2c}$$

The right-hand side of equation (4.2c) follows from the residue theorem.

The function $F(y)$ is bounded at the points $\pm\lambda_2$, $\pm\lambda_3$ and satisfies the following scalar R-H problem:

$$\begin{array}{lll} 1. & F(y) \text{ is analytic in } \mathbb{C}\setminus[-\lambda_3,\lambda_3], & \\ 2. & F_+(y)+F_-(y) = \frac{v_1}{y}, & y\in\Sigma, \\ 3. & F_+(y)-F_-(y) = \xi v_1, & y\in(-\lambda_2,\lambda_2), \\ 4. & F(y) = O(y^{-1}), & y\to\infty. \end{array} \tag{4.3}$$

Let us define

$$T(y) := e^{\left(-\frac{Nl}{2}\right)\sigma_3}Y(y)e^{-\left(Ng(y)-F(y)\right)\sigma_3}e^{\frac{Nl\sigma_3}{2}}, \tag{4.4}$$

where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. The matrix function $T(y)$ is the solution to the R-H problem

$$\begin{array}{lll} 1. & T(y) \text{ is analytic in } \mathbb{C}\setminus\mathbb{R} & \\ 2. & T_+(y) = T_-(y)J_T(y), & y\in\mathbb{R}, \\ 3. & T(y) = I + O(y^{-1}), & y\to\infty, \end{array} \tag{4.5}$$

where

$$J_T(y) := \begin{pmatrix} e^{-N\left(g_+(y)-g_-(y)\right)+F_+(y)-F_-(y)} & e^{-N\left(\tilde{g}_+(y)+\tilde{g}_-(y)\right)-F_+(y)-F_-(y)+\frac{v_1}{y}} \\ 0 & e^{N\left(g_+(y)-g_-(y)\right)-F_+(y)+F_-(y)} \end{pmatrix}, \quad y\in\mathbb{R} \tag{4.6}$$

and $\tilde{g}(y)$ is defined in (3.20b).

We now perform a standard technique in the steepest decent analysis (see [4, 8, 9]): we open lenses around the intervals Σ_j, $j=1,2$. The interiors $L_{\pm j}$ and contours $\Xi_{\pm j}$ of the lenses are defined as in figure 1.

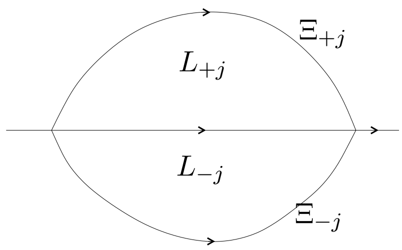

Figure 1: The opening of the lenses around the intervals Σ_j, $j = 1, 2$. $L_{\pm j}$ and $\Xi_{\pm j}$ are the interiors and contours of the lenses in the upper/lower half plane respectively. We also denote $\Xi_j = \Xi_{+j} \cup \Xi_{-j}$ and $\Xi = \Xi_1 \cup \Xi_2$, $j = 1, 2$.

Let us introduce the matrix function

$$S(y) := \begin{cases} T(y), & y \in \mathbb{C} \setminus (L_{+j} \cup L_{-j}), \\ T(y) \begin{pmatrix} 1 & 0 \\ -e^{2N\tilde{g}(y)+2F(y)-\frac{v_1}{y}} & 1 \end{pmatrix}, & y \in L_{+j}, \\ T(y) \begin{pmatrix} 1 & 0 \\ e^{2N\tilde{g}(y)+2F(y)-\frac{v_1}{y}} & 1 \end{pmatrix}, & y \in L_{-j}. \end{cases} \tag{4.7}$$

for $j = 1, 2$. It satisfies the R-H problem

$$\begin{array}{lll} 1. & S(y) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, & \\ 2. & S_+(y) = S_-(y) J_S(y), & y \in \mathbb{R}, \\ 3. & S(y) = I + O(y^{-1}), & y \to \infty. \end{array} \tag{4.8}$$

where $J_S(y)$ is the jump matrix

$$J_S(y) := \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2N\tilde{g}(y)+2F(y)-\frac{v_1}{y}} & 1 \end{pmatrix}, & y \in \Xi, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & y \in \Sigma, \\ \begin{pmatrix} 1 & e^{-2N\tilde{g}(y)-2F(y)+\frac{v_1}{y}} \\ 0 & 1 \end{pmatrix}, & x \in \mathbb{R} \setminus (-\lambda_3, \lambda_3), \\ \begin{pmatrix} e^{N\pi i+\xi v_1} & e^{-N\left(\tilde{g}_+(y)+\tilde{g}_-(y)\right)-F_+(y)-F_-(y)+\frac{v_1}{y}} \\ 0 & e^{-N\pi i-\xi v_1} \end{pmatrix}, & y \in (-\lambda_2, \lambda_2) \end{cases} \tag{4.9}$$

The conditions (3.30) imply that away from some small discs $D_{\pm\lambda_j}$ of radius δ centered at $\pm\lambda_j$, $j = 2, 3$, the off-diagonal entries of the jump matrix $J_S(y)$ are exponentially small in N, except on the intervals Σ_1 and Σ_2. This suggests the following approximation to the

R-H problem for $S(y)$:

$$\begin{array}{lll} 1. & S^\infty(y) \text{ is analytic in } \mathbb{C}\setminus[-\lambda_3,\lambda_3], & \\ 2. & S^\infty_+(y) = S^\infty_-(y)J^\infty(y), & y\in[-\lambda_3,\lambda_3], \\ 3. & S^\infty(y) = I + O(y^{-1}), & y\to\infty, \end{array} \tag{4.10}$$

where

$$J^\infty(y) := \begin{cases} \begin{pmatrix} e^{N\pi i+\xi v_1} & 0 \\ 0 & e^{-N\pi i-\xi v_1} \end{pmatrix}, & y\in(-\lambda_2,\lambda_2), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & y\in\Sigma. \end{cases} \tag{4.11}$$

The approximation $S^\infty(y)$ is known as *outer parametrix.*

4.2 The outer parametrix

Here and in the rest of §4, $\pm\lambda_j$ will always refer to the edge points of the support Σ of the equilibrium measure. Hence, $j=2,3$ only.

The solution to the R-H problem (4.10) exists and is uniformly bounded in N outside of small discs $D_{\pm\lambda_j}$ around the points $\pm\lambda_j$. Such a solution can be constructed in terms of elliptic theta functions as in Deift *et al* [9]. Here we follow their treatment.

Let $\mathcal{L}$ be the elliptic curve

$$q^2 = (y^2-\lambda_2^2)(y^2-\lambda_3^2) \tag{4.12}$$

and choose a canonical basis of cycles as in figure 2. Then, the holomorphic 1-form $\omega(y)$

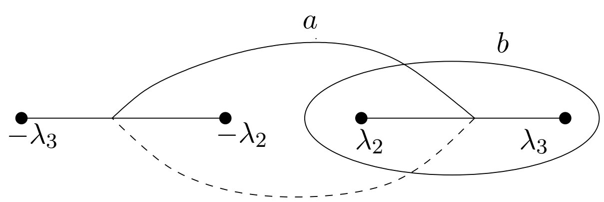

Figure 2: The a and b cycle of the elliptic curve (4.12).

dual to this set of cycles is given by

$$\omega(y) := \frac{dy}{K_0 q(y)}, \tag{4.13}$$

where K_0 was introduced in equation (4.2b). The Abel map is defined by

$$u(y) := \int_{\lambda_3}^{y} \frac{ds}{K_0 q(s)}. \tag{4.14}$$

where the contour of integration is chosen such that it does not intersect the interval $(-\infty, \lambda_3)$. Let Π be the b-period of the 1-form $\omega(y)$:

$$\Pi := 2\int_{\lambda_3}^{\lambda_2} \frac{dy}{K_0 q_+(y)}. \tag{4.15}$$

The elliptic theta function for the curve (4.12) with this choice of cycles is given by

$$\theta(s) := \sum_{m\in\mathbb{Z}} e^{i\pi\Pi m^2 + 2\pi i s m}, \tag{4.16}$$

Consider the function

$$\gamma := \left(\frac{(y-\lambda_2)(y+\lambda_3)}{(y+\lambda_2)(y-\lambda_3)}\right)^{\frac{1}{4}}, \tag{4.17}$$

where the arguments of the individual factors in the fourth root are chosen to be between $-\pi$ and π. Then the solution to (4.10) is given by

$$S^\infty(y) := H\begin{pmatrix} \frac{\gamma+\gamma^{-1}}{2}\frac{\theta\left(u(y)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}+d\right)}{\theta(u(y)+d)} & \frac{\gamma-\gamma^{-1}}{-2i}\frac{\theta\left(-u(y)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}+d\right)}{\theta(-u(y)+d)} \\ \frac{\gamma-\gamma^{-1}}{2i}\frac{\theta\left(u(y)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}-d\right)}{\theta(u(y)-d)} & \frac{\gamma+\gamma^{-1}}{2}\frac{\theta(-u(y)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}-d)}{\theta(u(y)+d)} \end{pmatrix}, \tag{4.18a}$$

$$H := \mathrm{diag}\left(\frac{\theta(u(\infty)+d)}{\theta\left(u(\infty)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}+d\right)}, \frac{\theta\left(u(\infty)+d\right)}{\theta\left(-u(\infty)-\frac{N}{2}-\frac{v_1\xi}{2\pi i}+d\right)}\right). \tag{4.18b}$$

where d is the constant

$$d := -\frac{1}{2}-\frac{\Pi}{2}+u_+(0) = -\frac{1}{2}-\frac{\Pi}{2}+\int_{\lambda_3}^{\lambda_2}\frac{ds}{K_0 q_+(s)}+\int_{\lambda_2}^{0}\frac{ds}{K_0 q_+(s)}. \tag{4.19}$$

Using the definition of b-period (4.15), of K_0 (4.2b) and the fact that $q(-s) = q(s)$ give $d = -\frac{1}{4}$.

4.3 Local parametrices near $\pm\lambda_2$ and $\pm\lambda_3$

Near the edge points $\pm\lambda_j$ the approximation of $S(y)$ by $S^\infty(y)$ fails. Therefore, we must solve the R-H problem (4.8) in small neighborhoods of these points and match the solutions to the outer parametrix (4.10) up to an error term of order $O(N^{-1})$. More precisely, let $\delta > 0$ and $D_{\pm\lambda_j}$ be a disc of radius δ centered at $\pm\lambda_j$. We would like to construct local parametrices $S^{(\pm\lambda_j)}(y)$ in $D_{\pm\lambda_j}$ such that

$$\begin{aligned} &1.\quad S^{(\pm\lambda_j)}(y) \text{ is analytic in } D_{\pm\lambda_j}\setminus\left(D_{\pm\lambda j}\cap(\mathbb{R}\cup\Xi)\right), \\ &2.\quad S_+^{(\pm\lambda_j)}(y) = S_-^{(\pm\lambda_j)}(z)J_S(y), \qquad y\in D_{\pm\lambda_j}\cap(\mathbb{R}\cup\Xi), \\ &3.\quad S^{(\pm\lambda_j)}(y) = \left(I+O(N^{-1})\right)S^\infty(y), \qquad y\in\partial D_{\pm\lambda_j}. \end{aligned} \tag{4.20}$$

These local parametrices are given by

$$S^{(\pm\lambda_j)}(y) := E_n^{(\pm\lambda_j)}(y)\Psi^{(\pm\lambda_j)}(\zeta_{\pm j})e^{\left(N\tilde{g}(y)+F(y)-\frac{v_1}{2y}\right)\sigma_3}. \tag{4.21}$$

The matrix functions $\Psi^{(\pm\lambda_j)}$ are constructed using Airy functions (see, *e.g.*, [7] pp. 213–216). The explicit expression of $\Psi^{(\pm\lambda_j)}$ is quite lengthy. Besides, it does not enter in our calculations. Therefore, we refer the interested reader to the original literature (see [4, 7, 8, 9]). The $\zeta_{\pm\lambda_j}$'s are conformal maps inside the neighborhoods $D_{\pm\lambda_j}$ given by the analytic continuation of the functions $\zeta^+_{\pm\lambda_j}$ to the whole $D_{\pm\lambda_j}$:

$$\zeta^+_{\pm\lambda_j} := \left(\frac{3}{2}N\right)^{\frac{2}{3}} \left(\tilde{g}(y) - \tilde{g}(\lambda_{\pm j})\right)^{\frac{2}{3}}, \quad \mathrm{Im}(y) > 0. \tag{4.22}$$

The matrix $E_n^{(\pm\lambda_j)}(y)$ is invertible and is the analytic continuation to the whole $D_{\pm\lambda_j}$ of the following quantity:

$$E_{n,+}^{(\pm\lambda_j)}(y) := \sqrt{\pi}e^{-\frac{i\pi}{12}}S_+^\infty(y)e^{\left(-F(y)+\frac{v_1}{2y}\right)\sigma_3}e^{\frac{i\pi}{4}\sigma_3}\begin{pmatrix} 1 & \mp(-1)^{j+1} \\ \pm(-1)^{j+1} & 1 \end{pmatrix}\left(\pm\zeta_{\pm\lambda_j}\right)^{\frac{\sigma_3}{4}}. \tag{4.23}$$

Remark 1. Since $E_n^{(\pm\lambda_j)}(y)$ is analytic inside $D_{\pm\lambda_j}$, we see that near the points $\pm\lambda_j$, the function $S^\infty(y)$ behaves like

$$\begin{aligned} S^\infty(y) \sim S_0^{(\pm\lambda_j)}(y)(y\mp\lambda_j)^{-\frac{\sigma_3}{4}} &\begin{pmatrix} 1 & \pm(-1)^{j+1} \\ \mp(-1)^{j+1} & 1 \end{pmatrix} \\ \times\, e^{-\frac{i\pi}{4}\sigma_3}e^{\left(-F(y)+\frac{v_1}{2y}\right)\sigma_3}&, \quad y \to \pm\lambda_j, \end{aligned} \tag{4.24}$$

where $S_0^{(\pm\lambda_j)}(y)$ is holomorphic and invertible at $\pm\lambda_j$.

4.4 The final transformation of the Riemann-Hilbert problem

We now show that the parametrices we constructed in §4.2 and §4.3 are indeed good approximations to the solution $S(y)$ of the R-H problem (4.8).

Let us define

$$R(y) := \begin{cases} S(y)\left(S^{(\pm\lambda_j)}(y)\right)^{-1}, & y \in D_{\pm\lambda_j}, \\ S(y)\left(S^\infty(y)\right)^{-1}, & y \in D_{\pm\lambda_j}. \end{cases} \tag{4.25}$$

Then the function $R(y)$ has jump discontinuities on the contour Γ_R shown in figure 3. In particular, $R(y)$ satisfies the R-H problem

$$\begin{array}{lll} 1. & R(y) \text{ is analytic in } \mathbb{C}\setminus\Gamma_R, & \\ 2. & R_+(y) = R_-(y)J_R(y), & y \in \Gamma_R, \\ 3. & R(y) = I + O(y^{-1}), & y \to \infty. \end{array} \tag{4.26}$$

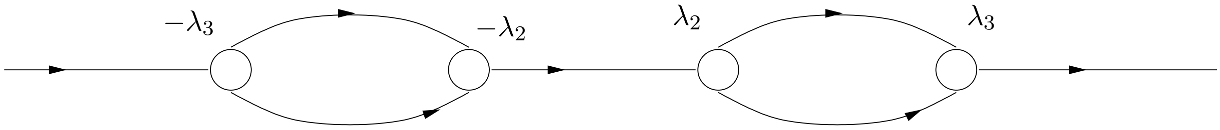

Figure 3: The contour Γ_R.

From the definition of $R(y)$ it follows that the jump matrix $J_R(y)$ has the following order of magnitude as $N \to \infty$:

$$J_R(y) = \begin{cases} I + O(N^{-1}), & y \in \partial D_{\pm\lambda_j}, \\ I + O\left(e^{-N\eta}\right), & \text{for some fixed } \eta > 0 \text{ for } y \in \Gamma_R \setminus D_{\pm\lambda_j}. \end{cases} \tag{4.27}$$

Then, using well established techniques (see,*e.g* [8] §7) we obtain

$$R(y) = I + O\left(\frac{1}{N\left(|y|+1\right)}\right), \tag{4.28}$$

uniformly in $\mathbb{C}$. Therefore, the solution $S(y)$ of the R-H problem (4.8) can be approximated by $S^{\infty}(y)$ and $S^{(\pm\lambda_j)}(y)$:

$$S(y) = \begin{cases} \left(I + O\left(N^{-1}\right)\right) S^{(\pm\lambda_j)}(y), & y \in D_{\pm\lambda_j}, \\ \left(I + O\left(N^{-1}\right)\right) S^{\infty}(y), & y \in \mathbb{C} \setminus D_{\pm\lambda_j}. \end{cases} \tag{4.29}$$

Combining these expressions with equations (4.7) and (4.4) we obtain an asymptotic formula for the solution of the original R-H problem (2.9), which can be inserted in the differential identities (2.10).

5 Asymptotics of the differential identities (2.10)

In this section we will compute an asymptotic formula for the partition function $G_N(v_1, v_2)$ defined in (2.3). The analysis is similar to those ones carried out in [12, 13, 14].

5.1 The finite v_2 regime

Consider the trace in the integrals (2.10):

$$\alpha(y) := \operatorname{Tr}\left(Y^{-1}(y)Y'(y)\sigma_3\right). \tag{5.1}$$

Since $\alpha(y)$ is analytic in a neighbourhood of the origin, in order to compute the integrals (2.10) we only need the first two terms in its Taylor expansion at $y = 0$. We have

$$\alpha(y) = \operatorname{Tr}\left(S^{-1}(y)S'(y)\sigma_3\right) + 2Ng'(y) - 2F'(y). \tag{5.2}$$

By using $S(y) = R(y)S^{\infty}(y)$ and the expression (4.18) for $S^{\infty}(y)$ we obtain

$$\alpha(y) = \mathrm{Tr}\left(\Theta^{-1}(y)\Theta'(y)\sigma_3\right) + 2Ng'(y) - 2F'(y) + O\left(\frac{1}{N}\right), \tag{5.3}$$

where

$$\Theta(y) = H^{-1}S^{\infty}(y). \tag{5.4}$$

It is convenient to split the computation of the right-hand side of this equation in two parts. We first determine

$$\alpha_0(y) := \mathrm{Tr}\left(\Theta^{-1}(y)\Theta'(y)\sigma_3\right), \tag{5.5}$$

then we evaluate

$$2Ng'(y) - 2F'(y). \tag{5.6}$$

Lemma 5.1. *We have*

$$\alpha_0(y) = \frac{M}{q(y)}, \tag{5.7}$$

where $q(y)$ is defined in (4.12) *and M is a constant.*

Proof. We first show that $\alpha_0(y)$ is analytic in $\mathbb{C} \setminus \Sigma$ and that

$$\alpha_{0,+}(y) = -\alpha_{0,-}(y), \quad y \in \Sigma. \tag{5.8}$$

By differentiating the jump conditions in (4.11), we see that $\Theta'(y)$ has the same jumps as $\Theta(y)$. Therefore, the quantity $\Theta^{-1}(y)\Theta'(y)$ has the discontinuity

$$\left(\Theta^{-1}(y)\Theta'(y)\right)_+ = \left(J^{\infty}(y)\right)^{-1}\left(\Theta^{-1}(y)\Theta'(y)\right)_- J^{\infty}(y), \quad y \in (-\lambda_3, \lambda_3) \tag{5.9}$$

and is analytic elsewhere. The jump discontinuities of $J^{\infty}(y)$ in (4.11) imply equation (5.8) and that

$$\alpha_{0,+}(x) = \alpha_{0,-}(y), \quad y \in (-\lambda_2, \lambda_2) \tag{5.10}$$

Therefore, $\alpha_0(y)$ must be equal to $q(y)$ multiplied by a rational function $r(y)$. Since $\alpha_0(x)$ is of order $O\left(y^{-2}\right)$ as $y \to \infty$ and is analytic away from the points $\pm\lambda_2$ and $\pm\lambda_3$, $r(y)$ can only have poles at the edge points $\pm\lambda_j$, $j = 2, 3$. From the behaviour of $S^{\infty}(y)$ near $\pm\lambda_j$ (see equation (4.24)), we see that $\alpha_0(y)$ can at worst have a singularity of order 1 at $\pm\lambda_j$. This implies that $r(y)$ is analytic at the points $\pm\lambda_j$. Hence, it must be a constant M. This proves the lemma. □

The constant M in (5.7) can be found from the asymptotic expansion of $\Theta(y)$ as $y \to \infty$.

Lemma 5.2. *The function $\alpha_0(y)$ in* (5.7) *is given by*

$$\alpha_0(y) = -\frac{2\pi i}{q(y)K_0\xi}\frac{\partial}{\partial v_1}\log\Big(\theta\big(u(\infty) + \varsigma - \tfrac{v_1\xi}{2\pi i}\big)\theta\big(u(\infty) - \varsigma + \tfrac{v_1\xi}{2\pi i}\big)\Big). \tag{5.11}$$

where $\varsigma = -\frac{N}{2} - \frac{1}{4}$.

Proof. From equations (4.10) and (5.4) the asymptotic expansion $\Theta(y)$ at $y=\infty$ has the form

$$\Theta(y) = H^{-1} + \frac{\Theta_1}{y} + O\left(y^{-2}\right), \quad y \to \infty, \tag{5.12}$$

where H is the constant in (4.18). Hence,

$$\alpha_0(y) = -\operatorname{Tr}\left(\frac{H\Theta_1}{y^2}\sigma_3\right) + O\left(y^{-3}\right), \quad y \to \infty. \tag{5.13}$$

The diagonal entries of Θ_1 are

$$(\Theta_1)_{11} = -\frac{H_{11}^{-1}}{K_0}\left(\frac{\theta'\big(u(\infty) - \frac{N}{2} - \frac{v_1\xi}{2\pi i} - \frac{1}{4}\big)}{\theta\big(u(\infty) - \frac{N}{2} - \frac{v_1\xi}{2\pi i} - \frac{1}{4}\big)} - \frac{\theta'\big(u(\infty) - \frac{1}{4}\big)}{\theta\big(u(\infty) - \frac{1}{4}\big)}\right) + \gamma_1 H_{11}^{-1}, \tag{5.14a}$$

$$(\Theta_1)_{22} = -\frac{H_{22}^{-1}}{K_0}\left(\frac{\theta'\big(u(\infty) + \frac{N}{2} + \frac{v_1\xi}{2\pi i} - \frac{1}{4}\big)}{\theta\big(u(\infty) + \frac{N}{2} + \frac{v_1\xi}{2\pi i} - \frac{1}{4}\big)} - \frac{\theta'\big(u(\infty) - \frac{1}{4}\big)}{\theta\big(u(\infty) - \frac{1}{4}\big)}\right) + \gamma_1 H_{22}^{-1}, \tag{5.14b}$$

where γ_1 is the coefficient of y^{-1} in the expansion of $\left(\gamma + \gamma^{-1}\right)/2$. Finally, equation (5.11) follows by substituting (5.14) into (5.13). □

We can now compute the logarithmic derivatives (2.10).

Proposition 5.1. *Let v_2 be of order $O(1)$ as $N \to \infty$ and define*

$$C := \frac{2v_1\lambda_2\lambda_3}{K_0}\int_{\lambda_2}^{-\lambda_2}\frac{ds}{s^2 q(s)}. \tag{5.15}$$

Then,

$$\begin{aligned}\frac{\partial \log G_N}{\partial v_1} &= \frac{\pi i\lambda_1^2}{v_2 K_0\xi}\frac{\partial}{\partial v_1}\log\Big(\theta\big(u(\infty) + \varsigma - \tfrac{v_1\xi}{2\pi i}\big)\theta\big(u(\infty) + \varsigma + \tfrac{v_1\xi}{2\pi i}\big)\Big) \\ &\quad - \frac{v_1}{2\lambda_1^2} - \frac{C}{2\lambda_2\lambda_3} + O\left(N^{-1}\right), \quad N \to \infty,\end{aligned} \tag{5.16a}$$

$$\frac{1}{N}\frac{\partial \log G_N}{\partial v_2} = N\left(\frac{1}{4} - \frac{v_2}{32}\left(\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_3^2}\right)^2 + \frac{8}{\lambda_1^4}\right)\right) + O\left(N^{-1}\right), \quad N \to \infty. \tag{5.16b}$$

Proof. We are only left to determine the term (5.6). First note that the derivative of $F(y)$ satisfies

$$\begin{aligned} &1. \quad F'(y) \text{ is analytic in } \mathbb{C}\setminus\Sigma, \\ &2. \quad F'_+(y) + F'_-(y) = -\frac{v_1}{y^2}, \qquad y \in \Sigma, \\ &3. \quad F'(y) = O(y^{-2}), \qquad y \to \infty. \end{aligned} \tag{5.17}$$

Moreover, since $F(y)$ is bounded at $\pm\lambda_j$, $j = 2, 3$, $F'(y)$ cannot have a singularity of order higher than 1 at these points. A function with these properties must be of the form

$$F'(y) = -\frac{v_1}{2y^2} - \frac{v_1\lambda_2\lambda_3 - \tilde{C}y^2}{2y^2\sqrt{(y^2-\lambda_2^2)(y^2-\lambda_3^2)}}, \tag{5.18}$$

for some constant $\tilde{C}$. This means that

$$\Omega(y) := \frac{v_1\lambda_2\lambda_3 - \tilde{C}y^2}{2y^2\sqrt{(y^2-\lambda_2^2)(y^2-\lambda_3^2)}}dy \tag{5.19}$$

is a meromorphic 1-form with a singularity at the origin such that $\Omega(y) \sim v_1 dy/2y^2$ as $y \to 0$ and is holomorphic elsewhere. To determine the constant $\tilde{C}$, note that the jump conditions (4.3) implies that the a-period of $\Omega(y)$ must vanish. This gives $\tilde{C} = C$, where C is defined in equation (5.15).

Therefore near $y = 0$ the function $\alpha(y)$ behaves as

$$\begin{aligned}\alpha(y) = N\Bigg(1 - \frac{v_2}{8}\left(\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_3^2}\right)^2 + \frac{8}{\lambda_1^4}\right)\Bigg)y + \frac{v_1}{\lambda_1^2} + \frac{C}{\lambda_2\lambda_3} \\ - \frac{2\pi i\lambda_1^2}{v_2K_0\xi}\frac{\partial}{\partial v_1}\log\Big(\theta\left(u(\infty) + \varsigma - \tfrac{v_1\xi}{2\pi i}\right)\theta\left(u(\infty) + \varsigma + \tfrac{v_1\xi}{2\pi i}\right)\Big) \\ + O(y^2) + O(N^{-1}).\end{aligned} \tag{5.20}$$

Finally, inserting the right-hand side into equations (2.10) gives formulae (5.16). □

5.2 The small v_2 regime

When v_2 becomes small, the local parametrices constructed in §4.3 must be defined in neighborhoods $D_{\pm\lambda_2}$ whose radii δ is smaller than λ_2. This affects the magnitude of the error term in the asymptotic formulae (5.16). We now study its effect. We shall see that by constructing local parametrices in shrinking neighborhoods of the points $\pm\lambda_2$, we can extend the validity of formulae (5.16) for $v_2 > N^{-3+\epsilon}$.

Since the error term is of order

$$O\left(\frac{|S^\infty(y)|^2\, e^{2|F(y)|+\frac{v_1}{|y|}}}{|\zeta_{\pm 2}|^{\frac{3}{2}}}\right) \tag{5.21}$$

on the boundary of $D_{\pm\lambda_2}$, we need to know the order of magnitude of $S^\infty(y)$ and of the conformal map $\zeta_{\pm\lambda_2}$.

Proposition 5.2. *The following asymptotic formulae as $v_2 \to 0$ hold.*

1. *The orders of magnitude of points λ_1 , λ_2 and λ_3 are*

$$\lambda_1 = O\left(v_2^{\frac{1}{3}}\right), \quad \lambda_2 = O\left(v_2^{\frac{1}{3}}\right) \quad \textit{and} \quad \lambda_3 = O(1). \tag{5.22a}$$

2. Let $D_{\pm\lambda_2}$ be small discs centered at $\pm\lambda_2$ with radii $\delta < \lambda_2$. Then, on the boundary of $D_{\pm\lambda_2}$ the conformal map $\zeta_{\pm\lambda_2}$ defined in (4.22) *is of order*

$$\zeta_{\pm\lambda_2} = O\left(N^{\frac{2}{3}} v_2^{-\frac{1}{9}} \delta\right). \tag{5.22b}$$

3. There exists a constant $k = O(1)$ as $v_2 \to 0$ such that the outer parametrix $S^{\infty}(y)$ defined in equations (4.18) *is of order*

$$S^{\infty}(y) = O\left(\left|\frac{\lambda_2}{\delta}\right|^{\frac{1}{4}} e^{k\frac{v_1}{\lambda_2}}\right). \tag{5.22c}$$

4. The order of magnitude of the exponential in equation (5.21) *is*

$$\exp\left(2\left|F(y)\right| + \tfrac{v_1}{|y|}\right) = O\left(e^{k\frac{v_1}{\lambda_2}}\right). \tag{5.22d}$$

Proof. Equation (5.22a) is an immediate consequence of formulae (3.13), (3.11) and (3.14). As $v_2 \to 0$ we obtain

$$\lambda_1 = (-2)^{-\frac{1}{6}} v_2^{\frac{1}{3}} + O\left(v_2\right), \tag{5.23a}$$

$$\lambda_2 = \left(-\frac{1}{2}\right)^{\frac{1}{2}} \lambda_1 + O\left(v_2\right), \tag{5.23b}$$

$$\lambda_3 = 2 + O\left(v_2^{\frac{2}{3}}\right). \tag{5.23c}$$

Next, consider the conformal maps $\zeta_{\pm\lambda_2}$. Inside $D_{\pm\lambda_2}$ they behave as follows:

$$\zeta_{\pm\lambda_2} = N^{\frac{2}{3}}\left(\varphi_{\pm}(y-\lambda_2) + O\left((y-\lambda_2)^2\right)\right), \tag{5.24}$$

where

$$\varphi_{\pm} := \pm\frac{(\lambda_2^2-\lambda_1^2)^{\frac{2}{3}}\left(\lambda_2\left(\lambda_2^2-\lambda_3^2\right)\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}\lambda_2^2}. \tag{5.25}$$

Hence, $\zeta_{\pm\lambda_2}$ is of order

$$\zeta_{\pm\lambda_2} = O\left(N^{\frac{2}{3}} v_2^{-\frac{1}{9}} \delta\right) \tag{5.26}$$

on the boundary of $D_{\pm\lambda_2}$.

Proving (5.22c) requires more work. Firstly, consider $\gamma \pm \gamma^{-1}$ in a small neighbourhood of λ_2 . If $|y-\lambda_2| = \delta$ is smaller than λ_2, we have

$$\begin{aligned}\gamma \pm \gamma^{-1} &= \left(\left(\frac{\delta}{2\lambda_2}\right)^{\frac{1}{4}}\left(\frac{\lambda_2+\lambda_3}{\lambda_2-\lambda_3}\right)^{\frac{1}{4}} \pm \left(\frac{2\lambda_2}{\delta}\right)^{\frac{1}{4}}\left(\frac{\lambda_2-\lambda_3}{\lambda_2+\lambda_3}\right)^{\frac{1}{4}}\right)\left(1+O\left(\frac{\delta}{|\lambda_2|}\right)\right) \\ &= \pm\left(\frac{2\lambda_2}{\delta}\right)^{\frac{1}{4}}\left(1+O\left(\frac{\delta}{|\lambda_2|}\right)+O\left(\lambda_2\right)\right).\end{aligned} \tag{5.27}$$

The case when $y = -\lambda_2$ can be treated analogously.

Let us now consider the theta functions that enter in the definition (4.18). First note that in the limit as $\lambda_2 \to 0$, the holomorphic 1-form $\omega(y)$ in (4.13) becomes a meromorphic 1-form with a simple pole at $y = 0$ with residue $1/(2\pi i)$ (see [1] and [14]). More explicitly

$$\omega(y) \to \frac{\lambda_3 dy}{2\pi y\sqrt{y^2 - \lambda_3^2}}, \quad \lambda_2 \to 0. \tag{5.28}$$

Furthermore, the constant K_0 has the limit

$$K_0 = \frac{2\pi}{\lambda_3} + O\left(\lambda_2^2\right), \quad \lambda_2 \to 0. \tag{5.29}$$

By writing the period Π as

$$\begin{aligned} \Pi &= 2\left(\int_{\lambda_3}^{\lambda_2^{\frac{1}{2}}} \frac{ds}{K_0 q_+(s)} + \int_{\lambda_2^{\frac{1}{2}}}^{\lambda_2} \frac{ds}{K_0 q_+(s)}\right) \\ &= 2\left(\int_{\lambda_3}^{\lambda_2^{\frac{1}{2}}} \frac{ds}{K_0 s\sqrt{s^2 - \lambda_3^2}}\big(1 + O\left(\lambda_2\right)\big) + \int_{\lambda_2^{\frac{1}{2}}}^{\lambda_2} \frac{ds}{K_0 i\lambda_3\sqrt{s^2 - \lambda_2^2}}\left(1 + O\left(\lambda_2\right)\right)\right), \end{aligned} \tag{5.30}$$

we can compute its limit:

$$\Pi = \left(\frac{1}{\pi i}\log|\lambda_2| - \frac{1}{\pi i}\log|16\lambda_3^2|\right)\big(1 + O\left(\lambda_2\right)\big), \quad \lambda_2 \to 0. \tag{5.31}$$

The Abel map $u(y)$ becomes

$$u(y) = \int_{\lambda_3}^{\lambda_2} \omega(s) + \int_{\lambda_2}^{y} \omega(s) = \frac{\Pi}{2} + \int_{\lambda_2}^{y} \omega(s). \tag{5.32}$$

Using this expression we obtain

$$\begin{aligned} u(y) &= \frac{\Pi}{2} + \left(\frac{\delta}{2\lambda_2}\right)^{\frac{1}{2}} \frac{2}{K_0\sqrt{\lambda_2^2 - \lambda_3^2}}\left(1 + O\left(\frac{\delta}{\lambda_2}\right) + O(\lambda_2)\right) \\ &= \frac{\Pi}{2} + \frac{1}{i\pi}\left(\frac{\delta}{2\lambda_2}\right)^{\frac{1}{2}}\left(1 + O\left(\frac{\delta}{\lambda_2}\right) + O(\lambda_2)\right), \quad \lambda_2 \to 0. \end{aligned} \tag{5.33}$$

We can now substitute equations (5.31) and (5.33) into the theta functions in (4.18) to obtain their orders of magnitude as $v_2 \to 0$. Let $\mathcal{A}$ be a constant vector that is independent of y and v_2. We have

$$\theta(s) = \theta\left(u(y) + \mathcal{A}\right) = \sum_{m \in \mathbb{Z}} e^{i\pi\Pi m^2 + 2i\pi(u(y)+\mathcal{A})m}. \tag{5.34}$$

The arguments of the exponentials become

$$m^2\Pi + 2\left(u(x)+\mathcal{A}\right)m = m(m+1)\Pi + 2\left(\frac{1}{\pi i}\left(\frac{\delta}{2\lambda_2}\right)^{\frac{1}{2}} + \mathcal{A}\right)m + O\left(\left(\frac{\delta}{\lambda_2}\right)^{\frac{3}{2}}\right) + O(\lambda_2), \quad \lambda_2 \to 0. \tag{5.35}$$

The asymptotic behaviour of the period Π in (5.31) gives

$$\theta\left(u(y)+\mathcal{A}\right) = 1 + e^{-\left(\frac{2\delta}{\lambda_2}\right)^{\frac{1}{2}} - 2\pi i\mathcal{A}}\left(1 + O\left(\frac{\delta}{\lambda_2}\right) + O(\lambda_2)\right), \quad \lambda_2 \to 0. \tag{5.36}$$

By substituting

$$\mathcal{A} = -\frac{1}{4} \quad \text{and} \quad \mathcal{A} = -\frac{N}{2} - \frac{v_1\xi}{2\pi i} - \frac{1}{4} \tag{5.37}$$

into (5.36), we see that the matrix elements in (4.18a) are bounded by infinity and zero. Note that, although the term $v_1\xi/(2\pi i)$ depends on v_2 through ξ, since our goal is to study the coefficients of v_1, we can always let v_1 be arbitrarily small so that the term $v_1\xi/(2\pi i)$ will only introduce a negligible error into (5.36). In particular, we have

$$\theta\left(u(y) - \tfrac{N}{2} - \tfrac{v_1\xi}{2\pi i} - \tfrac{1}{4}\right) = O\left(e^{k_1\frac{v_1}{\lambda_2}}\right), \tag{5.38}$$

where k_1 is of order $O(1)$ in v_2.

From equation (5.28) and the fact that $\lambda_3 = O(1)$ it follows that the term $u(\infty)$ in the constant H in (4.18b) remains finite as $v_2 \to 0$. In fact, we have

$$u(\infty) = \int_{\lambda_3}^{\infty} \frac{ds}{K_0 s\sqrt{s^2-\lambda_3^2}}\left(1 + O\left(\lambda_2^2\right)\right) = \frac{1}{4} + O\left(\lambda_2^2\right), \quad \lambda_2 \to 0. \tag{5.39}$$

Proceeding as in the derivation of (5.36), we see that also H remains finite and non-zero as $v_2 \to 0$.

Arguments similar to those ones that led to equation (5.33) give

$$|F(y)| = O\left(k_2\frac{v_1}{\lambda_2}\right) \quad \text{and} \quad \frac{v_1}{y} = O\left(k_3\frac{v_1}{\lambda_2}\right), \tag{5.40}$$

where $y = \pm\lambda_2 + \delta$, and k_2 and k_3 are of order $O(1)$ in v_2. Let us write

$$k := \max\left\{k_1, k_2, k_3\right\}. \tag{5.41}$$

Equation (5.22c) follows by combining formulae (5.27), (5.38) and (5.39), and equation (5.22d) is a simple consequence of(5.40). □

We are now ready to prove

Corollary 5.1. *Let $0 < \epsilon < 3$. Then, formulae (5.16) hold for $v_2 > N^{-3+\epsilon}$ with an error terms of order*

$$\mathcal{E} = O\left(N^{-\frac{\epsilon}{9}} e^{k\frac{v_1}{\lambda_2}}\right). \tag{5.42}$$

Proof. Combining equation (5.21) with the asymptotic formulae (5.22) gives

$$\mathcal{E} = O\left(\frac{v_2^{\frac{1}{3}} e^{k\frac{v_1}{\lambda_2}}}{N\delta^2}\right). \tag{5.43}$$

Then, equation (5.42) follows by setting $v_2 = O(N^{-3+\epsilon})$ and

$$\delta = O\left(N^{-1+\frac{2\epsilon}{9}}\right) = O\left(\lambda_2 N^{-\frac{\epsilon}{9}}\right). \tag{5.44}$$

□

6 Asymptotics of the ensemble average $E_N(z,t)$

We are now in a position to give an asymptotic formula for the ensemble average (1.22) and complete the proof of Theorem 1.1.

Let us translate formulae (5.16) back into the original variables $z = N\sqrt{v_2}$ and $t = \sqrt{N} v_1$. We obtain

$$\frac{\partial \log G_N}{\partial z} = z\left(\frac{1}{2} - \frac{v_2}{16}\left(\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_3^2}\right)^2 + \frac{8}{\lambda_1^4}\right)\right) + O(zN^{-1-\frac{\epsilon}{9}}), \tag{6.1a}$$

$$\begin{aligned}\frac{\partial \log G_N}{\partial t} &= \frac{\pi i \lambda_1^2}{v_2 K_0 \xi}\frac{\partial}{\partial t}\log\left(\theta\left(u(\infty) + \varsigma - \tfrac{t\xi}{2\pi i\sqrt{N}}\right)\theta\left(u(\infty) + \varsigma + \tfrac{t\xi}{2\pi i\sqrt{N}}\right)\right)\\ &\quad - \frac{t}{2N\lambda_1^2} - \frac{C}{2\lambda_2\lambda_3\sqrt{N}} + O\left(N^{-\frac{1}{2}-\frac{\epsilon}{9}}\right),\end{aligned} \tag{6.1b}$$

where the λ_j's, K_0 and ξ are functions of z only. Proposition 5.2 and Corollary 5.1 imply that these formulae hold only in the range $c_1 N^{-\frac{1}{2}} < z < c_2 N$, where c_1 and c_2 are constants independent of z and N.

Equation (5.15) implies that

$$C = \frac{v_1\lambda_2\lambda_3}{2K_0\pi} + O\left(\lambda_2^2\right), \quad \lambda_2 \to 0. \tag{6.2}$$

From this expression, the definition of ξ (4.2c) and the behaviour of the λ_j's as $v_2 \to 0$ (see equation (5.22a)), we obtain

$$\frac{\partial \log G_N}{\partial z} = \frac{z}{2} - \frac{3}{2^{\frac{4}{3}}} N^{\frac{2}{3}} z^{\frac{1}{3}} + O\left(\frac{z^{\frac{5}{3}}}{N^{\frac{2}{3}}}\right) + O\left(zN^{-1-\frac{\epsilon}{9}}\right), \tag{6.3a}$$

$$\frac{\partial \log G_N}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}\log\left(\theta\left(u(\infty)+\varsigma-\tfrac{t\xi}{2\pi i\sqrt{N}}\right)\theta\left(u(\infty)+\varsigma+\tfrac{t\xi}{2\pi i\sqrt{N}}\right)\right) + \left(\frac{tN^{\frac{1}{3}}}{2^{\frac{2}{3}} z^{\frac{4}{3}}} - \frac{t}{16N}\right)\left(1+O\left(v_2^{\frac{2}{3}}\right)\right) + O\left(N^{-1-\frac{\epsilon}{9}} e^{k\frac{v_1}{\lambda_2}}\right). \tag{6.3b}$$

Integrating these formulae gives

$$E_N(z,t) = B_N^{\epsilon} \exp\left(\frac{z^2}{4} - \frac{9}{2^{\frac{10}{3}}}\left(N^{\frac{2}{3}} z^{\frac{4}{3}} - N^{\frac{2\epsilon}{3}}\right) + \frac{t^2 N^{\frac{1}{3}}}{2^{\frac{5}{3}} z^{\frac{4}{3}}}\right) \times \left(\theta\left(u(\infty)+\varsigma-\tfrac{t\xi}{2\pi i\sqrt{N}}\right)\theta\left(u(\infty)+\varsigma+\tfrac{t\xi}{2\pi i\sqrt{N}}\right)\right)^{\frac{1}{2}} \times \left(1+O\left(\frac{z^{\frac{8}{3}}}{N^{\frac{2}{3}}}\right)+O\left(N^{-\frac{1}{2}+\frac{\epsilon}{9}} e^{k\frac{v_1}{\lambda_2}}\right)+O\left(\frac{t^2}{N}\right)\right), \tag{6.4}$$

where B_N^{ϵ} is the constant

$$B_N^{\epsilon} = \int_{\mathbb{R}^N} \left(\prod_{j=1}^{N} e^{-\frac{1}{2N^{-\epsilon} x_j^2}}\right) P_{\mathrm{GUE}}(x_1,\ldots,x_N) d^N x, \tag{6.5}$$

which is independent of t.

The error term $O\left(z^{\frac{8}{3}}/N^{\frac{2}{3}}\right)$ in equation (6.4) diverges unless $z = O\left(N^{1/4}\right)$. Therefore, we will restrict the validity of (6.4) to $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$.

We now use the expression of the period Π in equation (5.31) to simplify the theta functions in formula (6.4). Combining the definition of the theta function (4.16) and the limit of $u(\infty)$ as $\lambda_2 \to 0$ (5.39) gives

$$\theta\left(u(\infty)+\varsigma\pm\tfrac{t\xi}{2\pi i\sqrt{N}}\right) = 1 + \sum_{m\neq 0}\left(\frac{\lambda_2}{16\lambda_3^2} e^{c_N \lambda_2}\right)^{m^2} e^{2\pi i m\left(\frac{1}{4}+\varsigma\pm\frac{t\xi}{2\pi i\sqrt{N}}+O(\lambda_2^2)\right)}. \tag{6.6}$$

where c_N is a constant of order $O(1)$ in λ_2. Hence, we have

$$\theta\left(u(\infty)+\varsigma\pm\tfrac{t\xi}{2\pi i\sqrt{N}}\right) = 1 + \left(\frac{\lambda_2}{32} e^{c_N\lambda_2}\right)\cos\left(2\pi\left(\tfrac{1}{4}+\varsigma\pm\tfrac{t\xi}{2\pi i\sqrt{N}}\right)\right) + O\left(\lambda_2^3\right). \tag{6.7}$$

Substituting this formula back into (6.4) and letting $N\to\infty$ followed by $\epsilon\to 0$ gives

$$\lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{E_N(z,t)}{B_N^{\epsilon}}\exp\left(\frac{9}{2^{\frac{10}{3}}}\left(N^{\frac{2}{3}} z^{\frac{4}{3}} - N^{\frac{2\epsilon}{3}}\right) - \frac{t^2 N^{\frac{1}{3}}}{2^{\frac{5}{3}} z^{\frac{4}{3}}}\right) = \exp\left(\frac{z^2}{4}\right). \tag{6.8}$$

This limit completes the proof of Theorem 1.1.

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