# Mathematics 1EM/S (MATH 10600/10500) Weeks 1-12

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**Abstract**

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1 Algebra (Weeks 1 & 2)

1.1 Elementary operations

1.1.1 Addition, subtraction, multiplication and division

Associative law:
1. Of addition: \( a + (b + c) = (a + b) + c \).
2. Of multiplication: \( ab(c) = (ab)c \).

Examples:
1. \( 4 + (7 + 3) = 4 + 10 = 14 \) and \( (4 + 7) + 3 = 11 + 3 = 14 \).
2. \( 3 \cdot (2 \cdot 5) = 3 \cdot 10 = 30 \) and \( (3 \cdot 2) \cdot 5 = 6 \cdot 5 = 30 \).

Subtraction and division are not associative. Examples:
1. \( 7 - (5 - 2) = 4 \) but \( (7 - 5) - 2 = 0 \).
2. \( 8 \div (4/2) = 4 \) but \( (8/4)/2 = 1 \).

Commutative law:
1. Of addition: \( a + b = b + a \).
2. Of multiplication: \( ab = ba \).

Subtraction and division are not commutative. Examples:
1. \( 9 - 7 = 2 \) but \( 7 - 9 = -2 \).
2. \( 20 \div 10 = 2 \) but \( 10 \div 20 = \frac{1}{2} \).

Distributive law: Multiplication distributes over addition: \( a \cdot (b + c) = (a \cdot b) + (a \cdot c) \).

Example: \( 5 \cdot (3 + 4) = 5 \cdot 7 = 35 \) and \( (5 \cdot 3) + (5 \cdot 4) = 15 + 20 = 35 \).

Addition is not distributive over multiplication. Example: \( 5 + (3 \cdot 4) = 5 + 12 = 17 \) but \( (5 + 3) \cdot (5 + 4) = 8 \cdot 9 = 72 \).

1.1.2 Exponentiation

An exponent, also called index or power, is a number placed in a superscript position to the right to indicate repeated multiplication. Thus \( a^2 \) indicates \( a \cdot a \) and \( a^3 \) indicates \( a \cdot a \cdot a \).

Laws of indices:
1. \( a^m \cdot a^n = a^{m+n} \).
2. \( a^m/a^n = a^{m-n} \).
3. \( (a^m)^n = a^{mn} \).
4. \( a^{1/n} = \sqrt[n]{a} \).
5. \( a^0 = 1 \) when \( a \neq 0 \).
6. \( a^{-n} = 1/a^n \) (thus \( 1/a^{-n} = a^n \)).
7. \( a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \).
8. \( (a \cdot b)^n = a^n \cdot b^n \).
9. \( (\sqrt[n]{a})^m = \frac{a^m}{\sqrt[n]{m}} \).

Examples:
1. \( 4m^2 \cdot 5m^3 = 4 \cdot 5 \cdot m^2 \cdot m^3 = 20m^{2+3} = 20m^5 \).
2. \( \frac{36x^7}{2x^2} = \frac{36}{2} \cdot \frac{x^7}{x^2} = 12 \cdot x^{7-2} = 12x^5 \).
3. \( (5x^3)^3 = 5^3 \cdot (x^3)^3 = 125 \cdot x^{3 \cdot 3} = 125x^9 \).
4. \( 16^{1/2} = \sqrt{16} = 4 \).
5. \( 8^{1/3} = \sqrt[3]{8} = 2 \).
6. \( 13 \cdot 2^0 \cdot 2 = 13 \cdot 1 \cdot 2 = 26 \).
7. \( 5^{-3} = \frac{1}{5^3} = \frac{1}{125} \).
8. $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$.
9. $32^{-4/5} = \frac{1}{32^{4/5}} = \frac{1}{(\sqrt[5]{32})^4} = \frac{1}{16}$.

But:
1. $(a + b)^m \neq a^m + b^m$.
2. $(a - b)^m \neq a^m - b^m$.

Examples:
1. $(2 + 3)^2 = 5^2 = 25$, whereas $2^2 + 3^2 = 4 + 9 = 13$.
2. $(5 - 3)^2 = 2^2 = 4$, whereas $5^2 - 3^2 = 25 - 9 = 16$.

1.1.3 Expansions

When brackets are removed from expressions like $(a + b)(c + d)$, each term in the first bracket must multiply each term in the second bracket:

$$(a + b)(c + d) = ac + bc + ad + bd.$$  

Examples:
1. $n(2n + 3) = 2n^2 + 3n$.
2. $(n + 1)(n + 2) = n^2 + 2n + n + 2 = n^2 + 3n + 2$.

These are illustrated by the rectangles in Figure 1.

![Figure 1: $n(2n + 3)$ and $(n + 1)(n + 2)$](image)

The following special cases are known as binomial formulae:

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a - b)^2 = a^2 - 2ab + b^2$.
3. $(a + b)(a - b) = a^2 - b^2$.
4. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
5. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Examples:
1. $(4a + 7b)^2 = 16a^2 + 56ab + 49b^2$.
2. $(5x - yz)^2 = 25x^2 - 10xyz + y^2z^2$.
3. $(9m + 8)(8 - 9m) = (8 + 9m)(8 - 9m) = 64 - 81m^2$.
4. $(2x - 3y)^3 = (2x)^3 - 3 \cdot (2x)^2 \cdot 3y + 3 \cdot 2x \cdot (3y)^2 - (3y)^3$
   
   $= 8x^3 - 36x^2y + 54xy^2 - 27y^3$.
5. $(-x + 2y)^2 = (2y - x)^2 = 4y^2 - 4xy + x^2$.
6. $(-6a - c)^2 = -(6a + c)^2 = (6a + c)^2 = 36a^2 + 12ac + c^2$.

1.1.4 Simple factorisation

Factorisation is the reverse process of expansion. Examples:

1. $12n + 4 = 4 \cdot 3n + 4 \cdot 1 = 4(3n + 1)$.
2. $2n^2 - 6n = 2n(n - 3)$.
3. $32p^3 + 24p^3 + 16p^2 = 8p^2(4p^2 + 3p + 2)$.
4. $ap - 2aq + aq - 2bp = a(p + q) - 2b(q + p) = (p + q)(a - 2b)$.

Sometimes we can factorise by using the binomial formulae. Examples:

1. $x^2 - 25 = (x + 5)(x - 5)$.
2. $y^2 - 6y + 9 = (y - 3)^2$.
3. $a^2 + 20a + 100 = (a + 10)^2$. 

4
1.1.5 Algebraic fractions

Algebraic fractions have properties which are the same as those of numerical fractions. Examples:

1. \( \frac{x}{4} + \frac{2x}{3} = \frac{3x}{12} + \frac{8x}{12} = \frac{11x}{12} \).
2. \( \frac{5}{2x+y} - \frac{1}{4} = \frac{5 \cdot 4 - 2x+y}{2x+y \cdot 4} = \frac{20-2x+y}{8x+4y} \).
3. \( \frac{2}{x+4} - \frac{3}{x} \cdot \frac{x}{x^2-16} = \frac{2x^2-6x+12}{x(x+4)} = \frac{2x-6x+12}{x} = \frac{x-6}{x} \).

1.1.6 Changing the subject of formulae

Examples:

1. Solve for \( x \) the equation \( \sqrt{x^2-a^2} = a \).

\[ x^2 - a^2 = a^2 \implies x^2 = 2a^2 \implies x = \pm a \sqrt{2} \]

2. Solve for \( u \) the equation \( \frac{1}{u} + \frac{1}{v} = \frac{2}{f} \).

\[ \frac{1}{u} = \frac{2}{f} - \frac{1}{v} \implies \frac{1}{u} = \frac{2v-f}{fv} \implies f = (2v-f) \cdot u \implies u = \frac{fv}{2v-f} \]

3. Solve for \( a \) and \( b \) the equation \( T = \sqrt{\frac{Pbh}{4+a^2}} \).

\[ T^2 = \frac{Pbh}{4+a^2} \implies T^2 \cdot (4+a^2) = Pbh \implies b = \frac{T^2(4+a^2)}{Pbh} \]

\[ T^2 = \frac{Pbh}{4+a^2} \implies T^2 \cdot (4+a^2) = Pbh \implies 4+a^2 = \frac{Pbh}{T^2} \]

\[ \implies a^2 = \frac{Pbh}{T^2} - 4 \implies a = \pm \sqrt{\frac{Pbh}{T^2} - 4} \]

1.2 Linear expressions in one variable

1.2.1 Linear equations

Examples:

1. \( 5x + 7 = 37 \implies 5x = 30 \implies x = 6 \).

We check the answer by substituting \( x = 6 \) into the original equation: \( 5 \cdot 6 + 7 = 30 + 7 = 37 \).

2. \( \frac{7x - 1}{4} - \frac{1}{2} = \frac{4x + 2}{3} \)

\[ \frac{3 \cdot (7x - 1) - 6 \cdot 1}{12} = \frac{4 \cdot (4x + 2)}{12} \]

\[ \frac{21x - 3 - 6}{12} = \frac{16x + 8}{12} \]

\[ 21x - 9 = 16x + 8 \]

\[ 5x = 17 \]

\[ x = \frac{17}{5} \]

We check the answer by substituting \( x = \frac{17}{5} \) into the left hand side (LHS) and into the right hand side (RHS) of the equation and comparing both results.

LHS: \( \frac{7 \cdot \frac{17}{5} - 1}{2} = \frac{\frac{119}{5} - \frac{1}{2}}{2} = \frac{\frac{114}{5}}{2} - \frac{1}{2} = \frac{114}{10} \cdot \frac{1}{2} - \frac{1}{2} = \frac{57}{10} - \frac{5}{10} = \frac{52}{10} = \frac{26}{5} \).

RHS: \( \frac{\frac{53}{5} + \frac{26}{5}}{2} = \frac{\frac{79}{5}}{2} = \frac{79}{10} \cdot \frac{1}{2} = \frac{26}{5} \).

Since we obtain the same number both times our solution is correct.
3. The following does not appear at first to be a linear equation but can be reduced to one:

\[
(3x - 5)^2 - (2x + 3)^2 = 5(x - 2)(x + 2) - 2(14x + 3)
\]

\[
9x^2 - 30x + 25 - (4x^2 + 12x + 9) = 5(x^2 - 4) - 28x - 6
\]

\[
9x^2 - 30x + 25 - 4x^2 - 12x - 9 = 5x^2 - 20 - 28x - 6
\]

\[
5x^2 - 42x + 16 = 5x^2 - 28x - 26
\]

\[
x = -42
\]

We check the answer by substituting \( x = 3 \) into the left hand side and into the right hand side of the equation:

LHS: \((3 \cdot 3 - 5)^2 - (2 \cdot 3 + 3)^2 = (9 - 5)^2 - (6 + 3)^2 = 4^2 - 9^2 = 16 - 81 = -65\).

RHS: \(5(3 - 2)(3 + 2) - 2(14 \cdot 3 + 3) = 5 \cdot 1 \cdot 5 - 2(42 + 3) = -65\).

4. \[8x + 6 = 2^3x + 2 \cdot 3 \implies 8x + 6 = 8x + 6 \implies 0 = 0.\]

This is a true statement for any value of \( x \). So all real numbers solve this equation.

5. \[10x + 5 = 10x - 3 \implies 5 = -3.\]

This is a false statement for any value of \( x \). So there is no solution.

### 1.2.2 Linear inequalities

When both sides of an inequality are multiplied (or divided) by a negative number the sign reverses. *Examples*:

1. \[-2x > 8 \implies x < -4.\]

2. \[x - 1 < x + 2 \implies x < x + 3 \implies 0 < 3.\]

This is a true statement for any value of \( x \). So all real numbers solve this inequality.

3. \[3x - 3 \geq 4x - x + 6 \implies 3x \geq 3x + 9 \implies 0 \geq 9.\]

This is a false statement for any value of \( x \). So there is no solution.

### 1.2.3 Straight lines

The graph of the function \( f(x) = mx + c \) is a straight line. \( m \), which indicates the steepness of the line, is the slope. \( c \) is called "\( y \)-intercept" because it indicates that the line intersects the \( y \)-axis at \((0, c)\).

*Examples*:

1. The graph of the function \( f(x) = \frac{2}{3}x + 1 \) is shown in Figure 2.

2. Find the equation of the line which is parallel to the line above and passes through \((3, 4)\). Where does it intersect the axes?

A line which is parallel to the one above has slope \( \frac{2}{3} \). Thus it is of the form \( y = \frac{2}{3}x + c \). Since it passes through \((3, 4)\), when \( x = 3 \), \( y = 4 \). Thus \( 4 = \frac{2}{3} \cdot 3 + c \), i.e., \( c = 2 \). The desired equation is \( y = \frac{2}{3}x + c \).

When the line intersects the \( x \)-axis, \( y = 0 \). Then \( 0 = \frac{2}{3}x + 2 \), which gives \( x = -3 \).

When the line intersects the \( y \)-axis, \( x = 0 \). Then \( y = \frac{2}{3} \cdot 0 + 2 = 2 \).

3. Find the equation of the straight line which passes through \((-3, -2)\) and \((4, 1)\). Let the line be \( y = mx + c \). When \( x = -3 \), \( y = -2 \) and when \( x = 4 \), \( y = 1 \). Thus:

\[
-2 = m(-3) + c,
\]

\[
1 = m(4) + c.
\]

Subtracting the equations we obtain \(-3 = m(-7)\), i.e., \( m = \frac{3}{4} \). Substituting this in either equation gives \( c = -\frac{7}{4} \). Thus the desired equation is \( y = \frac{3}{4}x - \frac{7}{4} \).
1.3 Quadratic equations

Equations of the form \(ax^2 + bx + c = 0\) (with \(a \neq 0\)) are called quadratic equations. They can be solved by “completing the square”:

\[
\begin{align*}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
(x + \frac{b}{2a})^2 &= -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \\
x + \frac{b}{2a} &= \pm \sqrt{-\frac{4ac}{4a^2} + \frac{b^2}{4a^2}} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{align*}
\]

Example: Solve: \(4x^2 - 6x + 1 = 0\).

First we identify \(a, b,\) and \(c:\) \(a = 4, b = -6, c = 1.\) Then we replace \(a, b,\) and \(c\) in the formula by these numbers:

\[
x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 4 \cdot 1}}{2 \cdot 4} = \frac{6 \pm \sqrt{36 - 16}}{8} = \frac{6 \pm \sqrt{20}}{8} = \frac{6 \pm \sqrt{4 \cdot 5}}{8} = \frac{6 \pm 2\sqrt{5}}{8} = \frac{2(3 \pm \sqrt{5})}{8} = \frac{3 \pm \sqrt{5}}{4}.
\]

1.4 The factor and integer root theorems

The factor theorem and can be used to factorise a polynomial:

**Theorem 1 (Factor theorem)** For a polynomial \(f\), if there exists a constant \(k\) such that \(f(k) = 0\) then \((x - k)\) is a factor of \(f(x)\). Conversely, if \((x - k)\) is a factor of \(f(x)\), then \(f(k) = 0\).

Example: Determine if \(x - 3\) is a factor of \(f(x) = x^3 + 5x^2 - 17x - 21\).

We have to check if \(f(3) = 0\):

\[
f(3) = 3^3 + 5 \cdot 3^2 - 17 \cdot 3 - 21 = 27 + 45 - 51 - 21 = 0.
\]

Hence \(x - 3\) is a factor of the polynomial \(f(x) = x^3 + 5x^2 - 17x - 21\).

To factorise a polynomial whose coefficients are integers the integer root theorem is helpful:
Theorem 2 (Integer root theorem) If the coefficients of
\[ f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \]
are integers and the equation \( f(x) = 0 \) has any integral roots, then they are factors of the constant term \( a_0 \).

Example: Factorise \( f(x) = x^3 - 5x^2 + 2x + 8 \).

We search for integer roots by considering the factors of the constant term 8, i.e., ±1, ±2, ±4, ±8:
• \( f(1) = 1^3 - 5 \cdot 1^2 + 2 \cdot 1 + 8 = 6 \neq 0 \), therefore \((x - 1)\) is not a factor.
• \( f(-1) = (-1)^3 - 5 \cdot (-1)^2 + 2 \cdot (-1) + 8 = 0 \), therefore \((x + 1)\) is a factor.
• \( f(2) = 2^3 - 5 \cdot 2^2 + 2 \cdot 2 + 8 = 0 \), therefore \((x - 2)\) is a factor.

At this point we know that \( f(x) \) can be written as \((x + 1)(x - 2)(x - p)\). As \((+1)(-2)(-p)\) will give the constant term 8, we can conclude that \( p \) must be 4. Thus:
\[ f(x) = x^3 - 5x^2 + 2x + 8 = (x + 1)(x - 2)(x - 4). \]

2 Calculus (Weeks 3 & 4)

2.1 Functions

A function or map is a rule that assigns to every element \( x \) of a set \( X \) a unique element \( y \) of a set \( Y \), written as \( y = f(x) \), where \( f \) denotes the function. \( X \) is the domain and \( Y \) the codomain of the function. The set of all the elements \( f(x) \) is called the image or range of \( f \), and is denoted by \( f(X) \). It is a subset of the codomain. A function \( f \) can be represented graphically if \((x, f(x))\) is plotted on rectangular coordinate axes for \( x \) in \( X \).

Example: Plot the graph of the function \( f(x) = x^2 - 2x + 3 \).

First we try to find some ordered pairs \((x, f(x))\). Such pairs can be written as a table of values:

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

These pairs are plotted as points in a coordinate system and the points are connected, see Figure 3.

Figure 3: The graph of \( f(x) = x^2 - 2x + 3 \).
2.2 Derivatives

The derivative of a function $f$ at $x$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

(pronounced “f prime x”).

**Examples:**

1. $f(x) = x$.
   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h} = 1.$$

2. $f(x) = x^2$.
   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} 2x + h = 2x.$$

3. $f(x) = x^3$.
   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

It can be proved that:

- The derivative of a constant function is zero.
- If $f(x) = x^p$, where $p$ is a real number, then $f'(x) = p \cdot x^{p-1}$. Note that in such a case the function and the derivative are not always defined for all real numbers $x$.
- If $a$ is a real number and $f$ is differentiable at $x$, then $a \cdot f$ is also differentiable at $x$ and $(a \cdot f)'(x) = a \cdot f'(x)$.

**Examples:**

1. $f(x) = \sqrt{x} = x^{\frac{1}{2}}$.
   $$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

2. $f(x) = \frac{2}{x} = 2x^{-1}$.
   $$f'(x) = -6x^{-2} = -\frac{6}{x^2}.$$

2.3 Derivatives of sums, products and compositions

**Derivatives of sums:** If two functions $f$ and $g$ are differentiable at $x$, then $f + g$ is also differentiable at $x$ and $(f + g)'(x) = f'(x) + g'(x)$.

**Examples:**

1. $f(x) = 20x^3 + 9x^4$.
   $$f'(x) = 60x^2 + 36x^3.$$

2. $f(x) = \frac{9}{2x^2} - \frac{4}{7x^3} + \sqrt{2x} - \sqrt{3x^2} = \frac{9}{2}x^{-3} - \frac{4}{7}x^{-4} + x^{\frac{1}{2}} - x^{\frac{3}{2}}$.
   $$f'(x) = -\frac{45}{2}x^{-4} - \frac{36}{7}x^{-5} + \frac{1}{2}x^{\frac{-1}{2}} - \frac{4}{5}x^{\frac{-1}{2}} = -\frac{45}{2}x^{-4} - \frac{36}{7}x^{-5} + \frac{1}{3}x^{\frac{-1}{2}} - \frac{2}{5}x^{\frac{-1}{2}}.$$

**Derivatives of compositions (chain rule):** If $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$, then $f \circ g$ is differentiable at $x$ and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

**Examples:**

1. $f(x) = (2x - 5)^3$.
   $$f'(x) = 3(2x - 5)^2 \cdot 2 = 6(2x - 5)^2.$$

2. $f(x) = (7x - 20)^{12}$.
   $$f'(x) = 12(7x - 20)^{11} \cdot 7 = 84(7x - 20)^{11}.$$

3. $f(x) = \sqrt{5x^2 - 4x}$.
   $$f'(x) = \frac{1}{4}(5x^2 - 4x)^{-\frac{1}{2}} \cdot (10x - 4).$$
Derivatives of products: If \( f \) and \( g \) are differentiable at \( x \) then \( fg \) is also differentiable at \( x \) and
\[
(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x).
\]

**Examples:**
1. \( f(x) = x(x + 3)^4 \).
   \[
f'(x) = 1 \cdot (x + 3)^4 + x \cdot 4(x + 3)^3 = (x + 3)^3(x + 3 + 4x) = (5x + 3)(x + 3)^3.
   \]
2. \( f(x) = (x + 3)^2 \sqrt{2x^2 + 1} = (x + 3)^2(2x^2 + 1)^{\frac{1}{2}} \).
   \[
f'(x) = 2(x + 3) \cdot (2x^2 + 1)^{\frac{1}{2}} + (x + 3)^2 \cdot \frac{1}{2}(2x^2 + 1)^{-\frac{1}{2}} \cdot 4x
   = 2(x + 3) \left( \sqrt{2x^2 + 1} + \frac{(x + 3)x}{\sqrt{2x^2 + 1}} \right)
   = \frac{2(x + 3)(3x^2 + 3x + 1)}{\sqrt{2x^2 + 1}}.
\]

Derivatives of quotients: If \( f \) and \( g \) are differentiable at \( x \) and \( g(x) \neq 0 \) then \( \frac{f}{g} \) is also differentiable at \( x \) and
\[
\left( \frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.
\]

**Examples:**
1. \( f(x) = \frac{2x + 3}{1 - 5x} \).
   \[
f'(x) = \frac{2 \cdot (1 - 5x) - (2x + 3) \cdot (-5)}{(1 - 5x)^2} = \frac{2 - 10x + 10x + 15}{(1 - 5x)^2} = \frac{17}{(1 - 5x)^2}.
   \]
2. \( f(x) = \sqrt{\frac{5x}{4 - 3x}} \).
   \[
f'(x) = \frac{1}{2} \cdot \left( \frac{5x}{4 - 3x} \right)^{-\frac{1}{2}} \cdot \frac{5 \cdot (4 - 3x) - 5x \cdot (-3)}{(4 - 3x)^2}
   = \frac{1}{2} \sqrt{\frac{5x}{4 - 3x}} \cdot \frac{20 - 15x + 15x}{(4 - 3x)^2}
   = \frac{10 \cdot \sqrt{5x} \cdot (4 - 3x)}{\sqrt{5x} \cdot (4 - 3x)^2}
   = \frac{10}{\sqrt{5x} \cdot (4 - 3x)^{\frac{3}{2}}}.
\]

### 2.4 Tangents and normals

The **derivative** of a function \( f \) at \( x \) is also the slope of the tangent to the curve \( y = f(x) \) at the point \((x, f(x))\).

**Example:** Find the equation of the tangent at the point \((2, 6)\) to the curve \( y = f(x) = 2x^2 - 3x + 4 \).

The tangent is a straight line, so it can be written in the form \( y = mx + c \); we need to find \( m \) and \( c \). \( m \) can be found by evaluating the derivative at the point \((2, 6)\):
\[
f'(x) = 4x - 3,
\]
\[
m = f'(2) = 4 \cdot 2 - 3 = 5.
\]

So the equation of the tangent is \( y = 5x + c \). Now we have to find \( c \). As \((2, 6)\) is a point on the tangent we have:
\[
6 = 5 \cdot 2 + c \quad \Longrightarrow \quad c = -4.
\]

Thus the equation of the tangent at the point \((2, 6)\) is
\[
y = 5x - 4.
\]

The **normal** to a curve at a point is a straight line passing through the point which is perpendicular to the tangent at that point (Figure 4). If a line has slope \( m \neq 0 \), the slope of the line perpendicular to it is \(-\frac{1}{m}\). Two lines with gradients \( m_1, m_2 \neq 0 \) are perpendicular if \( m_1 m_2 = -1 \).

**Example:** Find the equation of the normal to the curve \( y = f(x) = 2(2 + x + x^2) \) at the point where \( x = -1 \).
First we simplify: 

\[ f(x) = 4 + 2x + 2x^2. \]

Then we find the point whose x-coordinate is \(-1\):

\[ f(-1) = 4 + 2 \cdot (-1) + 2 \cdot (-1)^2 = 4 - 2 + 2 = 4. \]

Hence the point is \((-1, 4)\). Then we determine \(f'(x)\):

\[ f'(x) = 2 + 4x. \]

The slope \(m_1\) of the tangent at \((-1, 4)\) is:

\[ m_1 = f'(-1) = 2 + 4(-1) = 2 - 4 = -2. \]

Therefore the slope \(m_2\) of the normal at \((-1, 4)\) is:

\[ m_2 = -\frac{1}{m_1} = -\frac{1}{-2} = \frac{1}{2}. \]

So we can write the equation of the normal is of the form:

\[ y = \frac{1}{2}x + c. \]

Since the normal passes through \((-1, 4)\), we have

\[ 4 = \frac{1}{2}(-1) + c \implies 4 = -\frac{1}{2} + c \implies c_2 = \frac{9}{2}. \]

Therefore the equation of the normal at \((-1, 4)\) is:

\[ y = \frac{1}{2}x + \frac{9}{2}. \]

### 2.5 Local extrema

Let \(f\) be a function.

1. \(x\) is a **local maximum** of \(f\) if for some \(\epsilon > 0\), \(f(x) \geq f(y)\) for all \(y\) between \(x - \epsilon\) and \(x + \epsilon\).
2. \(x\) is a **local minimum** of \(f\) if for some \(\epsilon > 0\), \(f(x) \leq f(y)\) for all \(y\) between \(x - \epsilon\) and \(x + \epsilon\).
3. \(x\) is a **local extremum** of \(f\) if it is either a local minimum or a local maximum of \(f\).

If \(x\) is a local extremum of \(f\) then \(f'(x) = 0\). The converse is false, i.e., it is possible that \(f'(x) = 0\) without \(x\) being an extremum of \(f\). Points where \(f'(x) = 0\) are called stationary points (regardless of whether they are local extrema or not).

**Example:**

1. \(f(x) = x^2\). \(f\) has a local minimum at 0: Since \(f'(x) = 2x\), \(f'(0) = 0\).
2. \(f(x) = -x^2\). \(f\) has a local maximum at 0: Since \(f'(x) = -2x\), \(f'(0) = 0\).
3. \(f(x) = x^3\). \(f\) has a stationary point, but not a local extremum, at 0: Since \(f'(x) = 3x^2\), \(f'(0) = 0\).

It is sometimes possible to distinguish between the stationary points by evaluating the second derivative of \(f\):
1. If $f'(x) = 0$ and $f''(x) < 0$ then $x$ is a local maximum of $f$.
2. If $f'(x) = 0$ and $f''(x) > 0$ then $x$ is a local minimum of $f$.

Note: If $f'(x) = 0$ and $f''(x) = 0$ then $x$ could be any of the three kinds of stationary points.

Example: Consider the functions $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$. 0 is a stationary point of all three functions and both the first and second derivatives of all three functions vanish at 0. However 0 is a local minimum for $f$, a local maximum for $g$ and neither for $h$.

Example: Identify and classify the stationary points of $f(x) = x^4 + 2x^3$.
First we solve $f'(x) = 0$ to find the stationary points of $f$: Since
$$f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3),$$
the stationary points are $x = 0$ and $x = -\frac{3}{2}$.
Next we evaluate $f''$ at these points: Since
$$f''(x) = 12x^2 + 12x = 12x(x + 1),$$
$f''(0) = 0$ and $f''(-\frac{3}{2}) = 9$.
Since $f''(-\frac{3}{2}) > 0$ we deduce that $-\frac{3}{2}$ is a local minimum of $f$. However since $f''(0) = 0$ we cannot conclude what kind of stationary point 0 is without further investigation of $f$.

The following two examples illustrate applications of these results:

Example: A rectangular enclosure is formed by using 1200m of fencing. Find the largest possible area that can be enclosed in this way and the corresponding dimensions of the rectangle.
Consider a rectangle with length $x$ (meters) and width $y$ (meters). The perimeter of such a rectangle is $2x + 2y$ which is given to be 1200. So we obtain:
$$x + y = 600.$$  
The area of such a rectangle is $\tilde{A}(x,y) = xy$. Using the equation above we can write this as a function of $x$ alone:
$$A(x) = x(600 - x).$$  
We differentiate $A$ twice to obtain
$$A'(x) = 600 - 2x,$$
$$A''(x) = -2.$$  
When $x$ maximises or minimises $A(x)$, $A'(x) = 0$. This gives us $x = 300$. Since $A''(300) < 0$ we conclude that $x = 300$ maximises $A(x)$. The corresponding $y$ is $600 - 300 = 300$, and the corresponding area is $\tilde{A}(300, 300) = 90,000$.
(Note that we can deduce a general result: For fixed perimeter, the rectangle with maximum area is a square.)

Example: A company that manufactures dog food wishes to pack the food in closed cylindrical tins. What should the dimensions of the tins be if each is to have a volume of 128π cm$^3$ and the minimum possible surface area?
Suppose each tin has radius $x$ (cm) and height $y$ (cm). The volume of such a tin is $\pi x^2 y$ which is given to be 128π. So we obtain:
$$x^2 y = 128.$$  
The surface area of such a tin is $\tilde{S}(x,y) = 2\pi x^2 + 2\pi xy$. Using the equation above we can write this as a function of $x$ alone:
$$S(x) = 2\pi x^2 + 2\pi x \frac{128}{x} = 2\pi x^2 + \frac{256\pi}{x}.$$  
We differentiate $S$ twice to obtain
$$S'(x) = 4\pi x - \frac{256\pi}{x^2},$$
$$S''(x) = 4\pi + \frac{512\pi}{x^3}.$$  
To find $x$ for which $S(x)$ is a minimum we first solve $S'(x) = 0$:
$$4\pi x - \frac{256\pi}{x^2} = 0 \implies 4\pi x^3 - 256\pi = 0 \implies x^3 = 64 \implies x = 4.$$  
Since $S''(4) = 12\pi > 0$ we conclude that $x = 4$ minimises $S(x)$. The corresponding $y$ is $\frac{128}{16} = 8$, and the corresponding surface area is $\tilde{S}(4,8) = 96\pi$. 12
3 Growth functions, exponentials & logarithms (Weeks 5 & 6)

3.1 Compound interest

*Example:* A person invests £1000 in a building society account which pays interest of 6% annually. Calculate the amount in the account over the next 3 years.

The interest in any year is 0.06 times the amount in the account at the beginning of the year. This is added on to the sum of money already in the account. The amount at the end of each year, after interest has been added, is 1.06 times the amount at the beginning of the year. So

Amount after 1 year = £1000 · 1.06 = £1060,
Amount after 2 years = £1060 · 1.06 = £1124,
Amount after 3 years = £1124 · 1.06 = £1191.

Continuing in this way, we obtain the following amounts to the nearest whole number of pounds:

<table>
<thead>
<tr>
<th>Number of years</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amount (£)</td>
<td>1000</td>
<td>1060</td>
<td>1124</td>
<td>1191</td>
</tr>
</tbody>
</table>

Notice that in the first year the interest is £60, but in the eighth year it is £67. This is because the amount on which the 6% is calculated has gone up from £1000 to £1124. Interest calculated this way is called *compound interest*. It follows the rule:

\[ T = P(1 + \frac{r}{100})^n \]

where \( T \) is the total of the initial amount plus interest, \( r \) is the rate of interest per year expressed as a percentage, \( n \) is the number of years, and \( P \) is the initial amount (invested or borrowed). In our example: \( r = 6 \) and \( P = 1000 \).

*Example:* If we invest £4000 at an annual rate of 6% compounded monthly, what will be the final value of the investment after 10 years?

In this case,

\[ T = P\left(1 + \frac{r}{100k}\right)^{nk} \]

where \( T, P, n \) and \( r \) are the same as before and \( k \) is the number of interest periods per year.

Since the interest is compounded monthly, there are 12 periods per year, so, \( k = 12 \). Thus we obtain

\[ T = 4000\left(1 + \frac{6}{100 \cdot 12}\right)^{120} = 4000\left(1 + 0.005\right)^{120} = 4000(1.005)^{120} \approx 7277.59. \]

After 10 years, the final value of the investment will be £7277.59.

3.2 Exponential growth/decay

In the limit as \( k \to \infty \) in the equation above we obtain exponential growth/decay:

\[ f(t) = ab^t \]

where \( a > 0 \) is the initial value (earlier we called this \( P \)), \( b > 0 \) is the growth factor, \( t \) is a real number and \( f(t) \) represents the quantity after \( t \) units of time (earlier we called this \( T \)). If \( b > 1 \), we deal with “exponential growth” and if \( 0 < b < 1 \) we deal with “exponential decay”.

*Example:* A cup of coffee at 85°C is placed in a freezer at 0°C. The temperature of the coffee decreases exponentially, so that after 5 minutes it is 30°C. What is the temperature after 3 minutes?

We use the formula \( f(t) = ab^t \). So far we know \( a = 85 \). To find \( b \) we use the information that after 5 minutes the temperature of the coffee is 30°C (\( f(5) = 30, \ a = 85, \ t = 5 \)):

\[ 30 = 85b^5 \implies b^5 = \frac{30}{85} \implies b = \sqrt[5]{\frac{30}{85}} \approx 0.811970902. \]

Now, we can answer the original question (\( a = 85, \ b = 0.811970902, \ t = 3 \)):

\[ f(3) = 85(0.811970902)^3 \approx 45.5. \]

Hence, after 3 minutes the cup of coffee has a temperature of about 45.5°C.
3.3 Exponential and logarithmic functions

Let $a > 0$ and $a \neq 1$. Then, the function defined by

$$f(x) = a^x$$

is called the exponential function with base $a$. If $a > 1$ then the function is increasing; if $a \in (0, 1)$ the function is decreasing (Figure 5).

The inverse of the exponential function with base $a$ is the logarithmic function with base $a$:

$$f(x) = \log_a x.$$  

(If $a = 10$ we refer to this as the common logarithm.) By “inverse” we mean that the following relations hold:

$$\log_a a^x = x \quad \text{and} \quad a^{\log_a x} = x.$$  

If $a > 1$ then the function is increasing; if $0 < a < 1$ it is decreasing (Figure 6).

Since exponential and logarithmic functions with the same base are inverse functions their graphs are reflected around the line $y = x$ (Figure 7).

**Examples:** Find $x$.

1. $\log_5 125 = 3$.  
   
   \[
   \log_5 125 = 3 \implies x^3 = 125 \implies x^3 = 5^3 \implies x = 5.
   \]

2. $\log_3 81 = x$.  
   
   \[
   \log_3 81 = x \implies 3^x = 81 \implies 3^x = 3^4 \implies x = 4.
   \]
3. \[ \log_3 x = 2. \]

It is easy to prove that logarithms obey the following laws (compare with §1.1.2):

1. \[ \log_a (xy) = \log_a x + \log_a y. \]
2. \[ \log_a \left( \frac{x}{y} \right) = \log_a x - \log_a y. \]
3. \[ \log_a x^n = n \cdot \log_a x. \]
4. \[ \log_a a = 1. \]
5. \[ \log_a 1 = 0. \]

**Examples:** The following logarithms are with respect to an arbitrary base:

1. Express \( \log(a^2bc) \) in terms of \( \log a, \log b \) and \( \log c. \)
   \[ \log(a^2bc) = \log a^2 + \log b + \log c = 2 \log a + \log b + \log c. \]
2. Express \( \log 2 - \log a + \frac{1}{2} \log b \) as a single logarithm.
   \[ \log 2 - \log a + \frac{1}{2} \log b = \log 2 - \log a + \log b^{\frac{1}{2}} = \log \frac{2\sqrt{b}}{a}. \]

A base of a logarithm can be “changed” as follows: Let \( y = \log_a a. \) Then:

\[ b^y = a \iff \log_c b^y = \log_c a \iff y \log_c b = \log_c a \iff y = \frac{\log_c a}{\log_c b}. \]

**Example:** \( \log_3 5 = \frac{\log_{10} 5}{\log_{10} 3}. \)

**Example:** A radioactive substance decays at a rate of 12% per hour.

1. After how many hours will half the radioactive material will be left?

We apply the formula for exponential growth and decay, \( f(t) = ab^r \): To find \( b \) we use the information we have, namely \( f(1) = 88, a = 100 \) and \( t = 1: \)

\[ 88 = 100b^1 \iff b = 0.88. \]

Next we evaluate \( t \) such that \( f(t) = 50: \)

\[ 50 = 100(0.88)^t \iff 0.88^t = 0.5 \iff t \log_{10} 0.88 = \log_{10} 0.5 \iff t = \frac{\log_{10} 0.5}{\log_{10} 0.88} \approx 5.42. \]

Hence half the radioactive material will be left after about 5.42 hours.

2. How many hours earlier did the substance have twice the current amount of radioactive material?

Again, we evaluate \( t \) such that \( f(t) = 200: \)

\[ 200 = 100 \cdot 0.88^t \iff 0.88^t = 2 \iff t \log_{10} 0.88 = \log_{10} 2 \iff t = \frac{\log_{10} 2}{\log_{10} 0.88} \approx -5.42. \]

Hence, about 5.42 hours earlier the substance had twice the current amount of radioactive material.

### 3.4 \( e \): Differentiating exponential and logarithmic functions

To find the derivative of an exponential function \( f(x) = a^x \) we use the definition:

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}. \]

The table below shows the values of \( \frac{a^h - 1}{h} \) (correct to four decimal places) for \( a = 2 \) and \( a = 3 \) as \( h \) decreases.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( h=0.1 )</th>
<th>( h=0.01 )</th>
<th>( h=0.001 )</th>
<th>( h=0.0001 )</th>
<th>( h=0.00001 )</th>
<th>( h=0.000001 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a=2 )</td>
<td>1</td>
<td>0.7177</td>
<td>0.6956</td>
<td>0.6934</td>
<td>0.6932</td>
<td>0.6931</td>
</tr>
<tr>
<td>( a=3 )</td>
<td>2</td>
<td>1.1612</td>
<td>1.1047</td>
<td>1.0992</td>
<td>1.0987</td>
<td>1.0986</td>
</tr>
</tbody>
</table>

This suggests that there exists an \( a \) between 2 and 3 for which

\[ \lim_{h \to 0} \frac{a^h - 1}{h} = 1. \]

This number is called \( e \); its value is 2.71828 (correct to 5 decimal places). When \( f(x) = e^x \) then \( f'(x) = e^x; \) this function is known as the exponential function.

**Examples:** Differentiate the following functions:
1. \( f(x) = e^{7x} \).
   \[ f'(x) = 7e^{7x} \]
2. \( f(x) = e^{x^2} \).
   \[ f'(x) = 2xe^{x^2} \]
3. \( f(x) = e^{5x^3} \).
   \[ f'(x) = 15x^2e^{5x^3} \]
4. \( f(x) = e^{\sqrt{x}} \).
   \[ f'(x) = \frac{1}{2\sqrt{x}}e^{\sqrt{x}} \]
5. \( f(x) = x^2 \cdot e^{-3x} \).
   \[ f'(x) = 2xe^{-3x} + x^2 \cdot (-3)e^{-3x} = e^{-3x}(2x - 3x^2) \]

The inverse (function) of the exponential function is the natural logarithm \( \ln x \), more commonly written as \( \log x \). (Note: \( \ln x = y \iff e^y = x \).) Its derivative can be deduced as follows: Let \( f(x) = \ln x \). Then:

\[
e^{f(x)} = e^{\ln x} = x.
\]

Taking the derivative on both sides we obtain:

\[
e^{f(x)}f'(x) = 1 \implies f'(x) = \frac{1}{e^{f(x)}} \implies f'(x) = \frac{1}{x}.
\]

**Examples:** Differentiate the following functions:

1. \( f(x) = \ln 2x \).
   \[ f'(x) = \frac{1}{2x} \cdot 2 = \frac{1}{x} \]
2. \( f(x) = \ln x^2 \).
   \[ f'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x} \]
3. \( f(x) = \ln \sqrt{4x - 5} = \ln(4x - 5)^{\frac{1}{2}} = \frac{1}{2} \ln(4x - 5) \)
   \[ f'(x) = \frac{1}{2} \cdot \frac{1}{4x - 5} \cdot 4 = \frac{2}{4x - 5} \]
4. \( f(x) = x \cdot (\ln x)^2 \).
   \[ f'(x) = 1 \cdot (\ln x)^2 \cdot x + x \cdot 2 \cdot \ln x \cdot \frac{1}{x} = (\ln x)^2 + 2 \ln x \]

How can we differentiate the function \( f(x) = a^x \) for some \( a > 0 \)? We see that

\[ \ln f(x) = \ln a^x = x \ln a. \]

Thus we can rewrite \( f \) as

\[ f(x) = e^{\ln f(x)} = e^{\ln a^x} = e^{x \ln a}. \]

Then

\[ f'(x) = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a. \]

We see that the derivative of an exponential function is proportional to the function itself.

**Examples:**

1. \( f(x) = 8^x - 3x \).
   \[ f'(x) = 8^x \cdot \ln 8 - 3 \]
2. \( f(x) = e^x + 21^x \).
   \[ f'(x) = e^x + 21^x \cdot \ln 21 \]

How can we differentiate the function \( f(x) = \log_a x \) for some \( a > 0 \)? We rewrite \( f \) as:

\[ f(x) = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x. \]

Then:

\[ f'(x) = \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \cdot \ln a}. \]

**Examples:**

1. \( f(x) = \log_2 \sqrt{x} \).
   \[ f'(x) = \frac{1}{\sqrt{x} \ln 2} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{\sqrt{x} \ln 2} \cdot \frac{1}{2} \sqrt{x} = \frac{1}{2x \ln 2} \]
2. \( f(x) = \log_{0.6} x \).
   \[ f'(x) = (-1) \cdot (\log_{0.6} x)^{-2} \cdot \frac{1}{x \ln 0.6} = - \frac{1}{x \ln 0.6 (\log_{0.6} x)^2} \]

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3.5 Aside: Inverse functions

We have seen that the exponential and logarithmic functions are inverses of each other. Here we explore the concept of the inverse of a function more carefully.

Let \( f : A \to B \). Then the function \( g : B \to A \) is called the \textit{inverse} of \( f \) if

\[
\forall x \in A, \quad g(f(x)) = x \quad \text{and} \quad \forall x \in B, \quad f(g(x)) = x.
\]

We refer to \( g \) as \( f^{-1} \) and to \( f \) as \( g^{-1} \).

\textbf{Example:} Find the inverse of the functions \( f(x) = 3x - 1 \) and \( g(x) = \frac{3}{x - 1} \).

We view the definition of \( f \) as an equation and solve for \( x \):

\[
f(x) = 3x - 1 \implies x = \frac{f(x) + 1}{3}.
\]

Thus, \( f^{-1}(x) = \frac{x + 1}{3} \). We can check our answer:

\[
f^{-1}(f(x)) = f^{-1}(3x - 1) = \frac{3x - 1 + 1}{3} = x,
\]

\[
f(f^{-1}(x)) = f\left(\frac{x + 1}{3}\right) = 3 \cdot \frac{x + 1}{3} - 1 = x.
\]

Similarly we view the definition of \( g \) as an equation and solve for \( x \):

\[
g(x) = \frac{3}{x - 1} \implies x = \frac{3}{g(x)} + 1.
\]

Thus \( g^{-1}(x) = \frac{3}{x} + 1 \).

For the inverse of a composition of functions,

\[
(f \circ g)^{-1} = g^{-1} \circ f^{-1}.
\]  

\textbf{Example:} Verify equation (1) for the functions \( f, g \) of the previous example.

First, we evaluate \( f \circ g \):

\[
(f \circ g)(x) = f(g(x)) = f\left(\frac{3}{x - 1}\right) = 3 \cdot \frac{3}{x - 1} - 1 = \frac{9 - (x - 1)}{x - 1} = \frac{10 - x}{x - 1}.
\]

Next we find its inverse:

\[
(f \circ g)(x) = \frac{10 - x}{x - 1}
\]

\[
(f \circ g)(x) \cdot (x - 1) = 10 - x
\]

\[x(f \circ g)(x) - (f \circ g)(x) = 10 - x
\]

\[x(f \circ g)(x) + x = 10 + (f \circ g)(x)
\]

\[x \cdot (f \circ g)(x) + 1 = 10 + (f \circ g)(x)
\]

\[x = \frac{10 + (f \circ g)(x)}{(f \circ g)(x) + 1}.
\]

Thus,

\[
(f \circ g)^{-1}(x) = \frac{10 + x}{x + 1}.
\]

Now we compare this with \( g^{-1} \circ f^{-1} \):

\[
(g^{-1} \circ f^{-1})(x) = g^{-1}(f^{-1}(x)) = g^{-1}\left(\frac{x + 1}{3}\right) = \frac{3}{x + 1} + 1 = \frac{9}{x + 1} + 1 = \frac{10 + x}{x + 1}.
\]

Thus we have verified that \( (f \circ g)^{-1} = g^{-1} \circ f^{-1} \).
4 Sequences & series (Week 7)

A sequence is a function defined on the positive integers. If \( u \) is a sequence then we write \( u_n = u(n) \) for the \( n \)th term of the sequence.

Examples:
1. \( u_n = n^2 \).
   1, 4, 9, 16, 25, . . .
2. \( u_n = \frac{n}{n+1} \).
   \( \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \).
3. \( u_1 = 99 \) and \( u_{n+1} = u_n - 2 \).
   99, 97, 95, 93, 91, . . .

(Descriptions like this are called recursive descriptions.)

The sum of the terms of a sequence is called a series. The sum of the first \( n \) terms is abbreviated
\[
\sum_{r=1}^{n} u_r = u_1 + u_2 + u_3 + \cdots + u_n.
\]

Examples:
1. \( \sum_{r=1}^{3} r = 1 + 2 + 3. \)
2. \( \sum_{r=4}^{7} (2r - 1) = 7 + 9 + 11 + 13. \)
3. \( \sum_{r=1}^{5} 1 = 1 + 1 + 1 + 1 + 1. \)
4. \( \sum_{r=3}^{8} r^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2. \)

4.1 Arithmetic sequences and series

4.1.1 Arithmetic sequences

An arithmetic sequence or arithmetic progression is a sequence in which each term (except the first) differs from the previous one by a constant amount, the common difference. If the first term is \( a \) and the common difference is \( d \), then the progression takes the form
\[ a, a + d, a + 2d, a + 3d, \ldots \]
and its \( n \)th term is:
\[ u_n = a + (n - 1)d. \]

Examples:
1. Write down the \( n \)th term of the arithmetic sequences that begin:
   (a) \( 17, 20, 23, \ldots \)
   \( u_n = 14 + 3n. \)
   (b) \( 1.3, 1.7, 2.1, \ldots \)
   \( u_n = 0.9 + 0.4n. \)
2. Find the number of terms in the arithmetic progression: 4, 5, 6, . . . , 17:
   For this sequence, \( a = 4 \) and \( d = 1 \). Let \( N \) be the number of terms. Then,
   \[ 4 + (N - 1) \cdot 1 = 17 \implies N - 1 = 13 \implies N = 14. \]
   The sequence has 14 terms.

4.1.2 Arithmetic series

A sum of the terms of an arithmetic progression is an arithmetic series. Writing the sum of the first \( n \) terms as \( S_n \), we have:
\[ S_n = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (a + (n - 2)d) + (a + (n - 1)d). \]

Rewriting these terms in the reverse order, we get:
\[ S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + 3d) + (a + 2d) + (a + d) + a. \]
Adding both equations we get:

\[
2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + (2a + (n - 1)d) + \cdots + (2a + (n - 1)d) \\
= n(2a + (n - 1)d).
\]

Thus:

\[
S_n = \frac{n}{2}(2a + (n - 1)d).
\]

The last term to be summed is the \(n^{th}\) term in the series, \((a + (n - 1)d)\). Writing this last term as \(l\),

\[
S_n = \frac{n}{2}(a + l).
\]

**Examples:**

1. Find the sum the first 12 terms of the arithmetic series 8 + 5 + 2 + \ldots.
   
   For this series, \(a = 8\), \(d = -3\), \(n = 12\). Thus:
   
   \[
   S_{12} = \frac{12}{2}(2 \cdot 8 + (12 - 1) \cdot (-3)) = 6 \cdot (-17) = -102.
   \]

2. Find the sum of the arithmetic series 5 + 7 + 9 + \ldots + 111.
   
   For this series, \(a = 5\), \(d = 2\), \(l = 111\). First we find the number \(n\) of terms:
   
   \[
   111 = 5 + (n - 1) \cdot 2 \implies 106 = (n - 1) \cdot 2 \implies 53 = n - 1 \implies n = 54.
   \]
   
   Next we find \(S_{54}\). Since \(S_n = \frac{n}{2}(a + l)\),
   
   \[
   S_{54} = \frac{54}{2} \cdot (5 + 111) = 27 \cdot 116 = 3132.
   \]

3. Find the first term and the common difference of the arithmetic sequence whose 4\(^{th}\) term is 15 and whose 9\(^{th}\) term is 35.
   
   \[
   u_4 = a + 3d = 15, \\
   u_9 = a + 8d = 35.
   \]
   
   Subtracting, \(5d = 20\), ie., \(d = 4\). This with either equation gives \(a = 3\). Thus, the first term is 3 and the common difference is 4.

### 4.2 Geometric sequences and series

#### 4.2.1 Geometric sequences

A *geometric sequence* or *geometric progression* is a sequence in which the ratio of each term (except the first) to the preceding term is a constant, the *common ratio*. If the first term is \(a\) and the common ratio is \(r\), then the sequence takes the form

\[
a, ar, ar^2, ar^3, \ldots
\]

and the \(n^{th}\) term is:

\[
u_n = ar^{n-1}.
\]

**Examples:**

1. Find the common ratio and the next two terms of the geometric sequence 2, -6, 18, -54.
   
   The common ratio is
   
   \[
r = \frac{-6}{2} = \frac{18}{-6} = \frac{-54}{18} = -3.
   \]
   
   Thus, \(u_5 = 2 \cdot (-3)^4 = 162\) and \(u_6 = 2 \cdot (-3)^5 = -486\).

2. Find the first term and common ratio of the geometric progression whose 2\(^{nd}\) term is 4 and whose 5\(^{th}\) term is 108.
   
   \[
u_2 = ar = 4, \\
   u_5 = ar^4 = 108.
   \]
   
   Dividing,
   
   \[
r^3 = \frac{108}{4} = 27 \implies r = 3.
   \]
   
   This with either equation gives \(a = \frac{4}{3}\). Thus the first term is \(\frac{4}{3}\) and the common ratio is 3.
4.2.2 Geometric series

A sum of the terms of a geometric progression is a geometric series:

\[ a + ar + ar^2 + ar^3 + \ldots \]

Writing the sum of the first \( n \) terms of an arithmetic series as \( S_n \), it follows that

\[ S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \]

Multiplying each term by \( r \),

\[ rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \]

Subtracting,

\[ S_n - rS_n = a - ar^n \quad \Rightarrow \quad S_n(1 - r) = a(1 - r^n) \quad \Rightarrow \quad S_n = a \frac{(1 - r^n)}{1 - r}. \]

**Examples:**

1. Find the sum of the first 5 terms of the geometric progression \( \frac{12}{25} + \frac{6}{5} + 3 + \ldots \).

   Here \( a = \frac{12}{25} \) and \( r = \frac{\frac{6}{5}}{\frac{12}{25}} = \frac{5}{2} \). Thus

   \[ S_5 = \frac{12 \left( \frac{5}{2} \right)^5 - 1}{\frac{5}{2} - 1} = 30.93. \]

2. Evaluate \( \sum_{i=1}^{8} 16 \cdot \left( \frac{1}{2} \right)^{i-1} \).

   We identify \( a = 16 \) and \( r = \frac{\frac{1}{2}}{16} = \frac{1}{2} \). Thus

   \[ S_8 = 16 \frac{1 - \left( \frac{1}{2} \right)^8}{1 - \frac{1}{2}} = 31.875. \]

3. The first term of a geometric series is 27 and its common ratio is \( \frac{4}{3} \). Find the least number of terms for which the series exceeds 550.

   Let us first suppose that \( S_n = 550 \). Then

   \[ 27 \left( \frac{4}{3} \right)^n - 1 = 550 \]

   \[ \left( \frac{4}{3} \right)^n - 1 = \frac{550}{27} \]

   \[ \left( \frac{4}{3} \right)^n = \frac{550}{27} \]

   \[ n \ln \left( \frac{4}{3} \right) = \ln \left( \frac{550}{27} \right) \]

   \[ n = \frac{\ln 550 - \ln 27}{\ln 4 - \ln 3} \approx 7.136. \]

   Thus for \( S_n > 550 \), we require \( n > 7.136 \), ie., \( n = 8 \).

   When \(-1 < r < 1\), ie., \( |r| < 1 \) consider the infinite geometric series

   \[ a + ar + ar^2 + ar^3 + ar^4 + \ldots \]

   We know that

   \[ S_n = a \frac{(1 - r^n)}{1 - r}. \]

   Since \( |r| < 1 \), \( r^n \to 0 \) as \( n \to \infty \). Thus \( S_n \to \frac{a}{1-r} \) as \( n \to \infty \). We say that the series is convergent and converges to

   \[ S_\infty = \lim_{n \to \infty} S_n = \frac{a}{1-r}. \]

   **Example:** Find the sum of the geometric series \( \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \ldots \):

   We identify: \( a = \frac{1}{2} \) and \( r = \frac{-\frac{1}{2}}{\frac{1}{2}} = -\frac{1}{2} \) (so \( |r| = \frac{1}{2} < 1 \)). Hence:

   \[ S_\infty = \frac{\frac{1}{2}}{1 - \left( -\frac{1}{2} \right)} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}. \]
5 Circles & pairs of straight lines (Week 8)

5.1 Circles

The points on a circle with centre \((a, b)\) and radius \(r > 0\) are precisely those points \((x, y)\) whose distance from \((a, b)\) is \(r\). Thus these points are precisely those which satisfy the equation:

\[(x - a)^2 + (y - b)^2 = r^2.\]

Points \((x, y)\) that lie outside this circle are precisely those whose distance from \((a, b)\) is greater than \(r\):

\[(x - a)^2 + (y - b)^2 > r^2;\]

and those inside the circle are precisely those whose distance from \((a, b)\) is less than \(r\):

\[(x - a)^2 + (y - b)^2 < r^2.\]

Examples:

1. Determine the centre and radius of the circle \((x - 1)^2 + (y + 3)^2 = 18\).

Comparing with the equation above shows that the centre is \((1, -3)\) and the radius is \(\sqrt{18} = 3\sqrt{2}\).

2. Find the equation of the circle with centre \((1, 2)\) and radius 3.

Substituting \(a = 1\), \(b = 2\) and \(r = 3\) in the equation above, we obtain:

\[(x - 1)^2 + (y - 2)^2 = 9\]

\[x^2 - 2x + 1 + y^2 - 4y + 4 = 9\]

\[x^2 + y^2 - 2x - 4y - 4 = 0.\]

3. Find the centre and radius of the circle \(x^2 + y^2 - 14x + 18y - 14 = 0\).

\[(x^2 - 14x) + (y^2 + 18y) = 14\]

\[(x^2 - 14x + 49) + (y^2 + 18y + 81) = 14 + 49 + 81\]

\[(x - 7)^2 + (y + 9)^2 = 144.\]

Hence, the circle has centre \((7, -9)\) and radius \(\sqrt{144} = 12\).

4. Do the points \((9, 1)\), \((8, 2)\) and \((7, 3)\) lie inside, on, or outside the circle \((x - 3)^2 + (y - 2)^2 = 25\)?

(a) \((9 - 3)^2 + (1 - 2)^2 = 37 > 25\). Hence \((9, 1)\) lies outside the circle.

(b) \((8 - 3)^2 + (2 - 2)^2 = 25\). Hence \((8, 2)\) lies on the circle.

(c) \((7 - 3)^2 + (3 - 2)^2 = 17 < 25\). Hence \((7, 3)\) lies inside the circle.

5.2 Pairs of straight lines (Simultaneous linear equations)

Example: Solve the following simultaneous equations:

1. \(2x + y = 7, 3x - 2y = 7\).

The first equation can be rewritten as: \(y = 7 - 2x\). Substituting this in the second equation we obtain:

\[3x - 2 \cdot (7 - 2x) = 7 \implies 3x - 14 + 4x = 7 \implies x = 3.\]

Then we substitute \(x = 3\) in one of the original equations to find \(y = 7 - 2 \cdot 3 = 1\). Thus the solution is \((3, 1)\). Geometrically we have found the intersection of the two straight lines that are described by these two equations (Figure 8).

2. \(3x + 2y = 12, 5x - 3y = 1\).

We multiply the first equation by 3 and the second equation by 2:

\[9x + 6y = 36,\]

\[10x - 6y = 2.\]

Now we add the equations together:

\[19x = 38 \implies x = 2.\]

Then we substitute \(x = 2\) in one of the original equations to find \(y\):

\[3 \cdot 2 + 2y = 12 \implies 2y = 6 \implies y = 3.\]
Figure 8: The straight lines $2x + y = 7$ and $3x - 2y = 7$.

3. $3x + 6y = 9$, $4x + 8y = 12$.
   We multiply the first equation by 4 and the second equation by 3:
   
   $12x + 24y = 36$,  
   $12x + 24y = 36$.

   We obtain the same equation! Thus these two equations describe the same straight line. Therefore we have an infinite number of solutions.

4. $3x + 6y = 9$, $4x + 8y = 2$. We multiply the first equation by 4 and the second equation by 3:
   
   $12x + 24y = 36$,  
   $12x + 24y = 6$.

   In other words we require $12x + 24y$ to be both 36 and 6, which is impossible. Thus there are no solutions. Geometrically (Figure 9) these two equations describe parallel straight lines (which never intersect).

Figure 9: The straight lines $3x + 6y = 9$ and $4x + 8y = 2$.

5.3 Circles and straight lines

Similar reasoning can be used to determine if a straight line cuts, touches (ie., is tangent to) or misses a circle.

**Example:** Does the straight line $3y = x + 5$ cut, touch or miss the circle $x^2 + y^2 - 6x - 2y - 15 = 0$?

The first equation gives $x = 3y - 5$. Substituting this in the second equation,

$(3y - 5)^2 + y^2 - 6(3y - 5) - 2y - 15 = 0$

$9y^2 - 30y + 25 + y^2 - 18y + 30 - 2y - 15 = 0$

$10y^2 - 50y + 40 = 0$

$y^2 - 5y + 4 = 0$.

This quadratic equation has either (i) no solutions, in which case the straight line misses the circle, or (ii) one solution, in which case the straight line touches (ie., is tangent to) the circle, or (iii) has two solutions in which case the straight line cuts the circle. Solving the equation we obtain,

$y = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 4}}{2} \implies y = 4$ or $y = 1$.

We find the corresponding $x$ by substituting these in the equation of the straight line: $x = 7$ and $x = -2$ respectively. Thus the straight line cuts the circle at $(7, 4)$ and $(-2, 1)$ (Figure 10).
Figure 10: The straight line $3y = x + 5$ and the circle $x^2 + y^2 - 6x - 2y - 15 = 0$.

6 Trigonometric functions (Weeks 9 - 11)

6.1 Radians, lengths of arcs and areas of sectors

Angles are commonly measured in degrees but it is mathematically natural to measure them in radians: An arc of length $\ell$ in a circle of radius $r$ subtends an angle of $\frac{\ell}{r}$ radians (Figure 11). Radians can be denoted by the superscript $c$. However, henceforth, unless specified otherwise, it is assumed that angles are measured in radians.

Figure 11: Definition of radians.

Since the circumference of a circle of radius $r$ is $2\pi r$ it follows that $2\pi^c = 360^o$. More generally,

$$\theta^o = \left(\frac{2\pi}{360}\right)^c \quad \text{and} \quad \theta^c = \left(\frac{360}{2\pi}\right)^o.$$

In particular:

<table>
<thead>
<tr>
<th>$\theta^o$</th>
<th>360</th>
<th>180</th>
<th>90</th>
<th>60</th>
<th>45</th>
<th>30</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta^c$</td>
<td>$\frac{2\pi}{180}$</td>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{\pi}{3}$</td>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{\pi}{6}$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Examples:

1. Convert $40^o$ to radians.

$$40^o = \left(40 \cdot \frac{\pi}{180}\right)^c = \frac{2}{9} \pi^c.$$

2. Express $\frac{5\pi}{6}^c$ in degrees.

$$\frac{5\pi}{6}^c = (\frac{5\pi}{6} \cdot \frac{180}{\pi})^o = 150^o.$$

3. Find the angle $\theta$ subtended by an arc of length 5 on a circle of radius 3.

$$\theta = \frac{5}{3}.$$

The area of a sector subtending an angle $\theta$ in a circle with radius $r$ is $\frac{1}{2} \theta r^2$. When $\theta = 2\pi$ we obtain the area of a circle, $\pi r^2$.

Example: Find the area $A$ of a sector subtending an angle 0.5 in a circle of radius 7.

$$A = \frac{1}{2} \cdot 0.5(7)^2 = 12.25.$$
6.2 The sine, cosine and tangent functions

With \( a, b \) and \( c \) being as in Figure 12, we define,

\[
\cos \beta = \frac{a}{c}, \quad \sin \beta = \frac{b}{c}, \quad \tan \beta = \frac{b}{a}.
\]  

(2)

It follows that \( \tan \beta = \frac{\sin \beta}{\cos \beta} \).

![Figure 12: A right-angled triangle.](image)

**Example:** In the triangle in Figure 12, let \( c = 110.6 \) and \( \alpha = 35.40^\circ \). Find \( \beta \), \( a \) and \( b \).

Since the sum of the angles in a triangle is \( 180^\circ \),

\( \beta = 180^\circ - 90^\circ - 35.40^\circ = 54.6^\circ \).

Next we evaluate \( a \):

\[
\sin \alpha = \frac{a}{c} \implies a = c \sin \alpha = 110.6 \cdot \sin 35.40 \approx 64.07.
\]

Of course, we get the same result when we use the other angle \( \beta \):

\[
\cos \beta = \frac{a}{c} \implies a = c \cos \beta = 110.6 \cos 54.6 \approx 64.07.
\]

The side \( b \) can now be evaluated by using the trigonometric functions again, but also by using Pythagoras’ theorem:

\[
a^2 + b^2 = c^2 \implies b = \sqrt{c^2 - a^2} = \sqrt{110.6^2 - 64.07^2} \approx 90.15.
\]

The graphs of the three trigonometric functions defined in (2) are shown in Figure 13. The graphs of these functions keep repeating themselves. Functions with this property are called periodic; the period of such a function is the smallest interval for which the function repeats itself. The period of the cosine and the sine function is therefore \( 2\pi \). Moreover,

\[
sin \theta = \sin(180^\circ - \theta), \quad \cos \theta = -\cos(180^\circ - \theta), \quad \tan \theta = -\tan(180^\circ - \theta),
\]

\[
sin \theta = -\sin(180^\circ + \theta), \quad \cos \theta = -\cos(180^\circ + \theta), \quad \tan \theta = \tan(180^\circ + \theta).
\]

In addition these functions are easy to evaluate for some angles: From Figure 14,

\[
\sin 0^\circ = 0, \quad \sin 30^\circ = \frac{1}{2}, \quad \sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \sin 90^\circ = 1,
\]

\[
\cos 0^\circ = 1, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 60^\circ = \frac{1}{2}, \quad \cos 0^\circ = 0,
\]

\[
\tan 0^\circ = 0, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}, \quad \tan 45^\circ = 1, \quad \tan 60^\circ = \sqrt{3}, \quad \tan 90^\circ = \infty.
\]

6.3 Trigonometric equations

There are three steps involved in solving equations such as \( \cos x = k \), \( \sin x = k \) and \( \tan x = k \):

1. Find \( \cos^{-1} k \), \( \sin^{-1} k \) or \( \tan^{-1} k \) using a calculator. (Note that the equations \( \cos x = k \) and \( \sin x = k \) have a solution only if \( -1 \leq k \leq 1 \).)

2. Find another root using:

\[
\cos x = \cos(360^\circ - x), \quad \sin x = \sin(180^\circ - x), \quad \tan x = \tan(180^\circ + x).
\]
3. Add or subtract multiples of $360^\circ$ to find the roots in the required interval.

**Examples:** Solve each equation for $0^\circ \leq x < 360^\circ$.

1. $\sin x = -0.84$. 
   Using a calculator, $\sin^{-1}(-0.84) \approx -57.146^\circ$, which is not in the required interval. Using $\sin x = \sin(180^\circ - x)$, another root is $180^\circ - (-57.146^\circ) \approx 237.14^\circ$, which is in the required interval. Adding $360^\circ$ to the first root, another root is $-57.146^\circ + 360^\circ \approx 302.86^\circ$, which is in the required interval. Thus the required roots are $237.14^\circ$ and $302.86^\circ$.

2. $\cos x = \frac{1}{2}$. 
   Using a calculator, $\cos^{-1}\left(\frac{1}{2}\right) \approx 70.528^\circ$, which is in the required interval. Using $\cos x = \cos(360^\circ - x)$, another root is $360^\circ - 70.528^\circ \approx 289.47^\circ$, which is also in the required interval. Thus the required roots are $70.528^\circ$ and $289.47^\circ$.

3. $\tan x = -2$. 
   Using a calculator, $\tan^{-1}(-2) \approx -63.434^\circ$, which is not in the required interval. Using $\tan x = \tan(180^\circ + x)$, another root is $180^\circ + (-63.434^\circ) \approx 116.57^\circ$, which is in the required interval. Adding $360^\circ$ to the first root, another root is $-63.434^\circ + 360^\circ \approx 296.57^\circ$, which is in the required interval. Thus the required roots are $116.57^\circ$ and $296.57^\circ$.

**Examples:** Solve each equation for $0^\circ \leq x < 360^\circ$.

1. $\sin \frac{x}{3} = \frac{\sqrt{3}}{2}$. 
   We require $0^\circ \leq \frac{x}{3} < 120^\circ$. We know that $\frac{x}{3} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ$, which is in the required interval. Using $\sin \frac{x}{3} = \sin(180^\circ - \frac{x}{3})$, another root is $180^\circ - 60^\circ = 120^\circ$, which is not in the required interval. By adding or subtracting $360^\circ$ we can’t find any further roots in the required interval. Thus the required root is $\frac{\sqrt{3}}{3} = 60^\circ$ ie., $x = 180^\circ$.

2. $\cos \frac{x}{4} = 0$. 
   We require $0^\circ \leq \frac{x}{4} < 72^\circ$. We know that $\frac{x}{4} = \cos^{-1}0 = 90^\circ$ which is not in the required interval. Using $\cos \frac{x}{4} = \cos(360^\circ - \frac{x}{4})$, another root is $360^\circ - 90^\circ = 270^\circ$, which is also not in the required interval. By adding or subtracting $360^\circ$ we can’t find any further roots in the required interval. Thus there are no solutions.

3. $\tan \frac{x}{2} = -3$. 
   We require $0^\circ \leq \frac{x}{2} < 180^\circ$. Using a calculator, $\frac{x}{2} = \tan^{-1}(-3) \approx -71.565^\circ$, which is not in the required interval. Using $\tan \frac{x}{2} = \tan(180^\circ + \frac{x}{2})$, another root is $180^\circ + (-71.565^\circ) \approx 108.43^\circ$ which is in the required
6.4 Trigonometric identities

We have already seen a relationship between trigonometric functions: From (2),
\[
\tan x = \frac{\sin x}{\cos x}.
\]

Here we shall see others. From Figure 12 and (2),
\[
\sin^2 x + \cos^2 x = \frac{b^2}{c^2} + \frac{a^2}{c^2} = \frac{b^2 + a^2}{c^2} = 1
\]
(using Pythagoras’ theorem). It is also true that:
\[
\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,
\]
\[
\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y.
\]

From which it follows that
\[
\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}.
\]

From these formulae we can deduce,
\[
\sin 2x = 2 \sin x \cos x,
\]
\[
\cos 2x = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1,
\]
\[
\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.
\]

The half-angle formulae follow from this:
\[
\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x),
\]
\[
\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x).
\]

Examples:

1. Prove the following identities.
   a) \(\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{2}{\sin x}\).
   b) \(\sin x + \frac{1 + \cos x}{\sin x} = \frac{\sin^2 x + (1 + \cos x)^2}{1 + \cos x}\).
   c) \(\frac{\sin^2 x + (1 + \cos x)^2}{1 + \cos x} = \frac{\sin^2 x + 1 + 2 \cos x + \cos^2 x}{1 + \cos x}\).

By adding or subtracting 360\(^\circ\) we can’t find any further roots in the required interval. Thus the required root corresponds to \(x = 108.43^\circ\) ie., \(x = 216.87^\circ\).

Examples: Solve each equation for \(0^\circ \leq x < 360^\circ\).

1. \(\sin 3x = -0.42\).
   We require \(0^\circ \leq 3x < 1080^\circ\). Using a calculator, \(3x = \sin^{-1}(-0.42) \approx -24.834^\circ\), which is not in the required interval. Using \(\sin(3x) = \sin(180^\circ - 3x)\), another root is \(3x = 180^\circ - (-24.834^\circ) \approx 204.83^\circ\), which is in the required interval. By adding multiples of 360\(^\circ\) to both roots we can find the further roots \(3x = 335.17^\circ, 564.83^\circ, 695.17^\circ, 924.83^\circ, 1055.17^\circ\) in the required interval. Dividing these roots by 3 we obtain \(x = 68.28^\circ, 111.72^\circ, 188.28^\circ, 231.72^\circ, 308.28^\circ, 351.72^\circ\).

2. \(\cos 2x = 0.246\).
   We require \(0^\circ \leq 2x < 720^\circ\). Using a calculator, \(2x = \cos^{-1}0.246 \approx 75.759^\circ\), which is in the required interval. Using \(\cos(2x) = \cos(360^\circ - 2x)\), another root is \(2x = 360^\circ - 75.759^\circ \approx 284.241^\circ\), which is also in the required interval. Adding 360\(^\circ\) to both roots we find the further roots \(2x = 435.76^\circ, 644.24^\circ\) in the required interval. Dividing these roots by 2 we obtain \(x = 37.88^\circ, 142.12^\circ, 217.88^\circ, 322.12^\circ\).

3. \(\tan 2x = 0\).
   We require \(0^\circ \leq 2x < 720^\circ\). We know, \(2x = \tan^{-1}0 = 0^\circ\), which is in the required interval. Using \(\tan x = \tan(180^\circ + x)\), another root is \(2x = 180^\circ + 0^\circ = 180^\circ\) which is also in the required interval. Adding 360\(^\circ\) to both roots we find the further roots 360\(^\circ\), 540\(^\circ\) which are in the required interval. Dividing these roots by 2 we obtain \(x = 0, 90^\circ, 180^\circ, 270^\circ\).
2. Simplify the following expressions.

(a) \((1 + \frac{\cos x}{\sin x} - \frac{1}{\sin x})(1 + \frac{\sin x}{\cos x} + \frac{1}{\cos x}) = 2.\)

\[
(1 + \frac{\cos x}{\sin x} - \frac{1}{\sin x})(1 + \frac{\sin x}{\cos x} + \frac{1}{\cos x}) = \frac{\sin x + \cos x - 1}{\sin x} \cdot \frac{\cos x + \sin x + 1}{\cos x} = \frac{(\sin x + \cos x)^2 - 1}{\sin x \cos x}
\]

\[
= \frac{\sin^2 x + 2 \sin x \cos x + \cos^2 x - 1}{\sin x \cos x} = \frac{2 \sin x \cos x}{\sin x \cos x} = 2.
\]

(c) \(\tan^2 x + \sin^2 x = (\frac{1}{\cos^2 x} + \cos x)(\frac{1}{\sin^2 x} - \cos x).\)

\[
\tan^2 x + \sin^2 x = \frac{\sin^2 x}{\cos^2 x} + \sin^2 x = \frac{1 - \cos^2 x}{\cos^2 x} + \sin^2 x = \frac{1}{\cos^2 x} - \sin^2 x = (\frac{1}{\cos x} + \cos x)(\frac{1}{\cos x} - \cos x).
\]

(d) \(\frac{1}{\tan^2 x} + \frac{1}{\tan^2 x} = \frac{1}{\sin^2 x} - \frac{1}{\sin^2 x}.\)

\[
\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{1}{\sin^4 x} - \frac{1}{\sin^4 x} \implies \frac{\cos^4 x + \cos^2 x}{\sin^4 x} = \frac{1}{\sin^4 x} - \frac{1}{\sin^4 x} \implies \frac{\cos^4 x + \cos^2 x}{\sin^4 x} = \frac{1 - \sin^2 x}{\sin^4 x} = \frac{\cos^2 x}{\sin^4 x} \implies \frac{\cos^2 x}{\sin^4 x} = \frac{\cos^2 x}{\sin^4 x}.
\]

2. Simplify the following expressions.

(a) \(\cos 4x \cos x - \sin 4x \sin x.\)

\[
\cos 4x \cos x - \sin 4x \sin x = \cos(4x + x) = \cos 5x.
\]

(b) \(2 \tan x \cos^2 x.\)

\[
2 \tan x \cos^2 x = 2 \frac{\sin x}{\cos x} \cdot \cos^2 x = 2 \sin x \cos x = \sin 2x.
\]

3. Solve the following equations for \(0^\circ \leq x < 360^\circ.\)

(a) \(4 - \sin x = 4 \cos^2 x.\)

\[
4 - \sin x = 4(1 - \sin^2 x) \implies 4 - \sin x = 4 - 4 \sin^2 x \implies 4 \sin^2 x - \sin x = 0
\]

\[
\implies \sin x \cdot (4 \sin x - 1) = 0 \implies \sin x = 0, \frac{1}{4}.
\]

There are four solutions in the required interval: \(0^\circ, 180^\circ, 14.48^\circ, 165.52^\circ.\)

(b) \(\sin^2 x + \cos x + 1 = 0.\)

\[
(1 - \cos^2 x) + \cos x + 1 = 0 \implies -\cos^2 x + \cos x + 2 = 0
\]

\[
\implies \cos x = \frac{-1 \pm \sqrt{1 + 8}}{-2} = -1, 2.
\]

As \(\cos x = 2\) is not possible, we get only one solution in the required interval: \(x = 180^\circ.\)

(c) \(3 \cos 2x + 2 + \cos x = 0.\)

\[
3(2 \cos^2 x - 1) + 2 + \cos x = 0 \implies 6 \cos^2 x + \cos x - 1 = 0
\]

\[
\implies \cos x = \frac{-1 \pm \sqrt{1 + 24}}{12} = \frac{1}{3}, -\frac{1}{2}.
\]

There are four solutions in the required interval: \(70.53^\circ, 120^\circ, 240^\circ, 289.47^\circ.\)

4. Obtain an expression for \(\sin 3x\) involving only powers of \(\sin x.\) If \(\sin x = \frac{1}{2}\) then the value of \(\sin 3x.\)

\[
\sin 3x = \sin(x + 2x) = \sin x \cos 2x + \cos x \sin 2x
\]

\[
= \sin x(1 - 2 \sin^2 x) + \cos x \cdot 2 \sin x \cos x = \sin x - 2 \sin^3 x + 2 \sin x \cos^2 x
\]

\[
= \sin x - 2 \sin^3 x + 2 \sin x(1 - \sin^2 x) = -4 \sin^3 x + 3 \sin x.
\]

If \(\sin x = \frac{1}{4}\) then,

\[
\sin 3x = -4 \cdot \left(\frac{1}{4}\right)^3 + 3 \cdot \frac{1}{4} = -\frac{1}{16} + \frac{12}{16} = \frac{11}{16}.
\]
6.5 Derivatives of trigonometric functions

The derivatives of sin and cos (when the angles are in radians) are:

\[
\begin{align*}
\sin(x)' &= \cos(x), \\
\cos(x)' &= -\sin(x).
\end{align*}
\]

**Examples:** Differentiate.

1. \(f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}\).
   
   \[f'(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot \sin(x)}{(\cos(x))^2} = \frac{1}{\cos^2(x)}.\]

2. \(f(x) = \frac{1}{\cos(x)} = (\cos(x))^{-1}.\)
   
   \[f'(x) = (-1) \cdot (\cos(x))^{-2} \cdot (-\sin(x)) = \frac{\sin(x)}{\cos^2(x)}.\]

3. \(f(x) = \cos(x^3 - 1).\)
   
   \[f'(x) = -\sin(x^3 - 1) \cdot 3x^2 = -3x^2\sin(x^3 - 1).\]

4. \(f(x) = \tan(7x).\)
   
   \[f'(x) = \frac{1}{\cos^2(7x)} \cdot 7 = \frac{7}{\cos^2(7x)}.\]

5. \(f(x) = \sin^2(x) \cos(3x).\)
   
   \[f'(x) = (2\sin(x) \cos(x)) \cdot \cos(3x) + \sin^2(x) \cdot (-\sin(3x)) \cdot 3 = \sin(x)(2\cos(x)\cos(3x) - 3\sin(x)\sin(3x)).\]

6. \(f(x) = \frac{\sin^3(x)}{\cos(3x)}.\)
   
   \[f'(x) = \frac{(4\sin^3(x) \cdot \cos(3x)) \cdot 3x - \sin^4(x) \cdot 6x}{\cos(3x)^2} = \frac{6\sin^3(x)}{3x^2} \cdot (12x\cos(3x) - \sin(3x)).\]

6.6 Aside: Implicit differentiation

An **implicit function** is one in which a relationship between, eg., two, variables is given, without one being explicitly defined in terms of the other. For example, \(y^2 - 3xy = x^2\), where \(y\) is implicitly defined as a function of \(x\) (or \(x\) is implicitly defined as a function of \(y\)).

**Implicit differentiation** is the differentiation of implicit functions. It is an application of the chain rule:

**Examples:** Find \(y'\) in terms of \(x\) and \(y\):

1. \(y^3 + 6x = x^2.\)
   
   \[3y^2 \cdot y' + 6 = 2x \implies y' = \frac{2x - 6}{3y^2}.\]

2. \(3y^2 + 2y + xy = x^3.\)
   
   \[6y \cdot y' + 2 \cdot y' + 1 \cdot y + x \cdot 1 \cdot y' = 3x^2 \implies 6yy' + 2y' + xy' = 3x^2 - y \implies y'(6y + 2 + x) = 3x^2 - y \implies y' = \frac{3x^2 - y}{6y + 2 + x}.\]

7 Complex Numbers (Week 12)

We extend (real) numbers by introducing an **imaginary** number \(i\) such that

\[i^2 = -1.\]

If we multiply a real number by \(i\), we call the result an **imaginary** number.

**Example:** \(\sqrt{-25} = \sqrt{25} \cdot (-1) = \sqrt{5^2} \cdot i^2 = \pm 5i.\)

If we add a real number and an imaginary number, the result is a complex number:

\[z = a + ib\]

where (here and below) \(a\) and \(b\) are real numbers. Complex numbers are added/subtracted by adding/subtracting their real parts and their imaginary parts. In order to multiply complex numbers, we “expand the brackets”.

**Examples:**
1. \((7 - 3i) + (4 + 9i) = 11 + 6i\).
2. \((7 - 3i) - (4 + 9i) = 3 - 12i\).
3. \((-2 + 3i)(4 - 8i) = -8 + 16i + 12i - 24i^2 = -8 + 28i = -8 + 28i\).

The conjugate of \(z = a + ib\) is \(\bar{z} = a - ib\).

The product \(z\bar{z}\) is a real number:

\[
z\bar{z} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 + b^2 = a^2 - (-1)b^2 = a^2 + b^2.
\]

Example:

\[
\frac{-2 + 3i}{4 - 8i} = \frac{-2 + 3i}{4 - 8i} \cdot \frac{4 + 8i}{4 + 8i} = \frac{-8 + 16i + 24i^2}{16 + 64} = \frac{-8 + 16i + 24 - 24}{80} = \frac{-2}{5} - \frac{1}{20}i.
\]

A quadratic equation always has complex solutions.

Example: \(x^2 - 6x + 25 = 0\).

\[
x = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.
\]

As complex numbers have two components, they can be represented geometrically on a plane referred to as the complex plane. A diagram showing complex numbers is said to be an Argand diagram. Each complex number is shown as a point on the Argand diagram (Figure 15).

Figure 15: Argand diagram

The argument of a complex number is the angle between the positive real axis and the arrow representing the complex number on the Argand diagram. From Figure 15,

\[
\sin(\arg(a + ib)) = \frac{b}{\sqrt{a^2 + b^2}},
\cos(\arg(a + ib)) = \frac{a}{\sqrt{a^2 + b^2}},
\tan(\arg(a + ib)) = \frac{b}{a}.
\]

Therefore a complex number can also be written in the form

\[
a + ib = \sqrt{a^2 + b^2} \cos(\arg z) + i \sin(\arg z)\;;
\]

\[
z = |z| (\cos(\arg z) + i \sin(\arg z)).
\]

This form is referred to as polar form.

Examples: Write in polar form by finding the modulus and argument:
1. $1 + i$.

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

$$\arg z = \tan^{-1} \left( \frac{1}{1} \right) = 45^\circ;$$

$$z = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ).$$

2. $-3 + 4i$.

$$|z| = \sqrt{(-3)^2 + 4^2} = 5,$$

$$\arg z = \tan^{-1} \left( \frac{4}{-3} \right) \approx 126.87^\circ;$$

$$z = 5(\cos 126.87^\circ + i \sin 126.87^\circ).$$

3. $-1 - i\sqrt{3}$.

$$|z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2,$$

$$\arg z = \tan^{-1} \left( \frac{-\sqrt{3}}{-1} \right) = 240^\circ;$$

$$z = 2(\cos 240^\circ + i \sin 240^\circ).$$

4. $i$.

$$|z| = \sqrt{0^2 + 1^2} = 1,$$

$$\arg z = \tan^{-1} \infty = 90^\circ;$$

$$z = 1(\cos 90^\circ + i \sin 90^\circ).$$