Representation Theory: Solutions 2

1.Let $\phi : U \to V$ be an isomorphism. If V is reducible, then there is an FG-submodule $W < V$ with $\{0\} \neq W \neq V$. But then $\phi^{-1}(W)$ is an FG-submodule of U with $\{0\} \neq \phi^{-1}(W) \neq U$. [$\phi^{-1}(W)$ is an FG-submodule since, for $g \in G$ and $w \in W$,

$$\phi^{-1}(w)g = \phi^{-1}(wg) \in \phi^{-1}(W).]$$

2. Let g be the generator of G, so $G = \{e, g\}$. Since V is 2-dimensional, every FG-submodule, apart from $\{0\}$ and V, must be 1-dimensional. Suppose $w = \lambda.e + \mu.g$ spans a 1-dimensional submodule. Then

$$wg = \mu.e + \lambda.g$$

must be a scalar multiple of w. Since $w \neq 0$, **either** $\lambda \neq 0$, in which case $\mu/\lambda = \lambda/\mu$ (and so $\mu \neq 0$ as well), and so $\mu = \pm\lambda$, **or** $\mu \neq 0$, in which case we also get $\mu = \pm\lambda$.

If the characteristic of F is not two, then, up to a scalar multiple, we have two possibilities, $w = e+g$ and $w = e-g$, and so there are two 1-dimensional submodules.

If the characteristic of F is two, then, up to a scalar multiple, there is only one possibility, $w = e+g = e-g$, and so only one 1-dimensional submodule.

3.(a) We need to show that W is closed under multiplication by x.

$$(e-x).x = x - x^2 = -(e-x) + (e-x^2) \in W,$$

and

$$(e-x^2).x = x - e = -(e-x) \in W,$$

so W is a submodule.

(b) Since W is 2-dimensional, to decide whether it is reducible we just need to decide whether it has a 1-dimensional submodule. Consider the 1-dimensional subspace V spanned by

$$0 \neq v = \lambda(e-x) + \mu(e-x^2) = (\lambda+\mu).1 - \lambda.x - \mu.x^2 \in W.$$

For V to be a submodule, we need $vx \in V$, so

$$vx = -\mu.e + (\lambda+\mu).x - \lambda.x^2$$

must be a scalar multiple αv of v, for some $\alpha \in F$.

Since $v \neq 0$, so no two of $\{\lambda+\mu, -\lambda, -\mu\}$ can be zero, we have

$$\alpha = \frac{(\lambda+\mu)}{-\mu} = \frac{-\lambda}{\lambda+\mu} = \frac{-\mu}{-\lambda},$$

and so

$$\alpha^3 = \frac{(\lambda+\mu)(-\lambda)(-\mu)}{(-\mu)(\lambda+\mu)(-\lambda)} = 1.$$

If $F = \mathbb{R}$, the only possibility is $\alpha = 1$, but then the three expressions for α easily lead to a contradiction. However, if $F = \mathbb{C}$, then there is a solution

$$\alpha = e^{2\pi i/3}, \lambda = 1, \mu = e^{2\pi i/3},$$

giving a 1-dimensional submodule (we get another 1-dimensional submodule if we use $e^{-2\pi i/3}$ in place of $e^{2\pi i/3}$).

4.(a) Let $v \in V^G$ and $g \in G$. Then $vg = v \in V^G$, so V^G is an FG-submodule.

(b) Let $\phi : V \to W$ be an isomorphism of FG-modules. Then the restriction of ϕ to V^G is an isomorphism $V^G \to W^G$. [If $v \in V^G$, and $g \in G$, then $\phi(v)g = \phi(vg) = \phi(v)$, so $\phi(v) \in W^G$, and similarly, if $w \in W^G$ then $\phi^{-1}(w) \in V^G$.]

(c) For $h \in G$ and $v \in V$,

$$\alpha(vh) = \sum_{g\in G} vhg = \sum_{g'\in G} vg'h = \alpha(v)h,$$

since as g runs over all elements of G, so does $g' = hgh^{-1}$. Also,

$$\alpha(v)h = \sum_{g\in G} vgh = \sum_{g''\in G} vg'' = \alpha(v),$$

since as g runs over all elements of G, so does $g'' = gh$, and so $\alpha(v) \in V^G$.

(d) No. For example, if $|G|$ divides the characteristic of F, and V is the trivial FG-module, then for $v \in V$, $\alpha(v) = |G|.v = 0$, so α is the zero map.

(e) Yes. In fact, whenever $|G|$ does not divide the characteristic of F, α is surjective, since if $v \in V^G$ then $v = \alpha(\frac{v}{|G|})$.

(f) Let $G = C_2$, generated by an element g, and let V be the 1-dimensional $\mathbb{C}G$-module on which g acts by $vg = -v$ for all $v \in V$.

5. No. Let $\{v_1, v_2, v_3, v_4\}$ be the obvious basis for V; then V^G is 2-dimensional, spanned by $v_1 + v_2$ and $v_3 + v_4$. However, if W is the regular FG-module, then W^G is 1-dimensional, spanned by $\sum_{g\in G} g$.