Random Matrices and the Riemann Zeta-Function: a Review

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Abstract

The past few years have seen the emergence of compelling evidence for a connection between the zeros of the Riemann zeta function and the eigenvalues of random matrices. This hints at a link between the distribution of the prime numbers, which is governed by the Riemann zeros, and the properties of waves in complex systems (e.g. waves in random media, or in geometries where the ray dynamics is chaotic), which may be modelled using random matrix theory. These developments have led to a significant deepening of our understanding of some of the most important problems relating to the zeta function and its kin, and have stimulated new avenues of research in random matrix theory. In particular, it would appear that several long-standing questions concerning the distribution of values taken by the zeta function on the line where the Riemann Hypothesis places its zeros can be answered using techniques developed in the study of random matrices.

1 Random matrices and the Riemann zeros

Linear wave theories may expressed in terms of matrices. Therefore, just as in complex (e.g. chaotic) dynamical systems, where statistical properties of the trajectories may be calculated by averaging with respect to an appropriate measure on phase space, statistical properties of the waves in complex systems may be calculated by averaging over ensembles of random matrices. In this sense, *Random matrix theory* [26] is to wave theories what statistical mechanics is to dynamical systems. In quantum mechanics, where the waves obey the Schrödinger equation, it was developed in the 1950s and 1960s by Wigner, Dyson and others and has been applied to a wide range of problems including systems with a large number of degrees of freedom (e.g. nuclei), systems in which the classical trajectories are chaotic (e.g. atoms, molecules, microelectronic devices), and systems in which the potential is random (e.g. disordered systems). It has found similar applications in acoustics, elasticity, and optics. (For an up-to-date review of the literature, see [15].)

The statistical properties of the eigenvalues and eigenfunctions of self-adjoint operators (e.g. the Schrödinger operator) can be modelled using the corresponding statistics for hermitian matrices; treating the real and imaginary parts of the matrix elements as independently distributed gaussian random variables. In the same way, unitary matrices can be used to model unitary operators (e.g. Green functions). In this case, one can use the fact that $N \times N$ unitary matrices form a compact group – the unitary group $U(N)$ – which comes with a natural invariant (uniform) measure: Haar measure.

My purpose here is to describe some rather surprising connections between the theory of waves in complex systems and number theory. Specifically, these connections concern random matrix theory and the Riemann zeta function $\zeta(s)$, which is central to the theory of the primes. I will focus on one aspect of this story: the connection between the distribution of values taken by the characteristic polynomials of random unitary matrices and that of the Riemann zeta function on the critical line, $s = 1/2 + it$, where the Riemann Hypothesis places its non-trivial (complex) zeros. This has led, conjecturally, as I shall explain, to a general solution to the long-standing problem of determining the moments of $\zeta(1/2 + it)$.

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \tag{1}$$

for Re$s > 1$, where p labels the primes, and then by analytic continuation to the rest of the complex plane. It has a single simple pole at $s = 1$, zeros at $s = -2, -4, -6$, etc., and infinitely many zeros, called the *non-trivial zeros*, in the *critical strip* $0 <$ Re$s < 1$. The Riemann Hypothesis states that all of the non-trivial zeros lie on the *critical line* Re$s = 1/2$; that is, $\zeta(1/2 + it) = 0$ has non-trivial solutions only when $t = t_n \in \mathbb{R}$ [30]. This is known to be true for at least 40% of the non-trivial zeros [8], for the first 100 billion of them [32], and for batches lying much higher [28].

In the following, for ease of presentation, we will assume the Riemann Hypothesis to be true, although this is not strictly necessary.

The mean density of the non-trivial zeros increases logarithmically with height t up the critical line. Specifically, the *unfolded* zeros

$$w_n = t_n \frac{1}{2\pi} \log \frac{t_n}{2\pi} \tag{2}$$

satisfy

$$\lim_{W \to \infty} \frac{1}{W} \# \{w_n \in [0, W]\} = 1; \tag{3}$$

that is, the mean of $w_{n+1} - w_n$ is 1. The question then arises as to the statistical distribution of the unfolded zeros: are they equally spaced, with unit spacing between neighbours, randomly distributed with unit mean spacing, or do they have some other distribution?

It is in this context that the connection with random matrices arises. Let A be an $N \times N$ unitary matrix; that is, $A \in U(N)$. Denote the eigenvalues of A by $\exp(i\theta_n)$, where $1 \leq n \leq N$ and $\theta_n \in \mathbb{R}$. Clearly the eigenphases θ_n have mean density $N/2\pi$, so the unfolded eigenphases

$$\phi_n = \theta_n \frac{N}{2\pi}, \tag{4}$$

have unit mean density (i.e. $\phi_n \in [0, N)$). Next, let us define

$$F(\alpha, \beta; A) = \frac{1}{N} \#\{\phi_n, \phi_m : \alpha \leq \phi_n - \phi_m < \beta\}. \tag{5}$$

The key step now is to average $F(\alpha, \beta; A)$ over A, chosen uniformly with respect to (normalized) Haar measure on $U(N)$. This average will be denoted by

$$F_U(\alpha, \beta; N) = \int_{U(N)} F(\alpha, \beta; A) dA. \tag{6}$$

Dyson proved in 1963 that

$$F_U(\alpha, \beta) = \lim_{N \to \infty} F_U(\alpha, \beta; N) \tag{7}$$

exists and takes the form

$$F_U(\alpha, \beta) = \int_\alpha^\beta \left[1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 + \delta(x) \right] dx, \tag{8}$$

where $\delta(x)$ is Dirac's δ-function [26]. The integrand in (8) may be thought of as the *two-point correlation function* for the eigenphases of a random unitary matrix, unfolded to have unit mean spacing. The fact that it is a non-trivial function of correlation distance x means that eigenphases are correlated in a non-trivial way.

The connection between the pair correlation of the Riemann zeros, as measured by

$$F_\zeta(\alpha, \beta) = \lim_{W \to \infty} \frac{1}{W} \#\{w_n, w_m \in [0, W] : \alpha \leq w_n - w_m < \beta\}, \tag{9}$$

and that of random matrix eigenvalues was made in 1973 by Montgomery [27], who conjectured that

$$F_\zeta(\alpha, \beta) = F_U(\alpha, \beta). \tag{10}$$

This conjecture has turned out to be extremely influential.

In his original paper, Montgomery proved a theorem which provides substantial support for his conjecture (10). The fourier transform of the two-point correlation function (i.e. of the integrand in (8)) may easily be calculated to be

$$k_U(\tau) = \int_{-\infty}^{\infty} \left[1 - \left(\frac{\sin(\pi x)}{\pi x} \right)^2 + \delta(x) \right] \exp(2\pi i x \tau) dx - \delta(\tau) = \begin{cases} |\tau| & |\tau| < 1 \\ 1 & |\tau| \geq 1 \end{cases}. \tag{11}$$

Montgomery showed that it follows from the prime number theorem (which states that the number of primes less than X grows asymptotically like $X/\log X$ as $X \to \infty$) that the analogue of $k_U(\tau)$

for the Riemann zeros coincides with the expression on the right of (11) in the range $|\tau| < 1$. His conjecture thus boils down to the claim that it coincides with the expression on the right in the range $|\tau| \geq 1$ as well.

There is substantial evidence in support of Montgomery's conjecture. First, Odlyzko has computed the two-point correlation function numerically for batches of zeros high up on the critical line (e.g. near to the 10^{20}th zero) and his results are in striking agreement with it [28]; see, for example, Figure 1. Second, the conjectured form for the fourier transform of the two-point correlation function in the range $|\tau| \geq 1$ may be shown to follow, heuristically, from an asymptotic analysis based on a conjecture of Hardy and Littlewood concerning correlations between the primes [22].

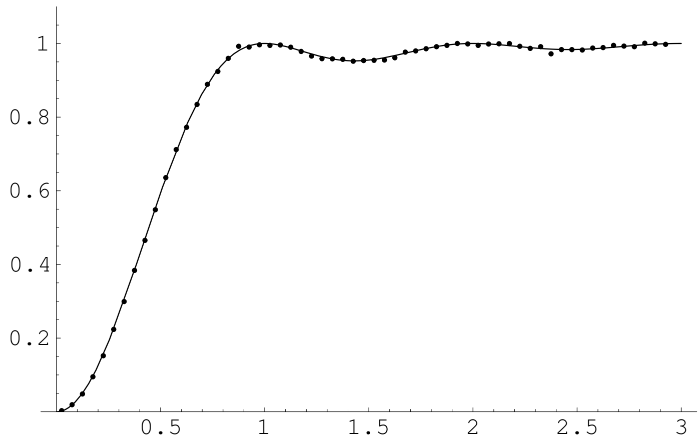

Figure 1: The two-point correlation function of 10^6 Riemann zeros around the height of the 10^{20}th zero (dots) and of the eigenphases of random unitary matrices in the limit as $N \to \infty$ (smooth curve). (Figure courtesy of AM Odlyzko.)

Montgomery's conjecture generalizes immediately to relate correlations between n-tuples of zeros and the corresponding correlations between n-tuples of eigenphases. His theorem also generalizes, that is the fourier transforms of the n-point correlation functions for the zeros and random-matrix eigenphases coincide in appropriately restricted ranges [29]. Asymptotic calculations based on prime correlations again support the conjecture outside these ranges [5, 6]. Odlyzko has computed various statistical measures of the zero distribution which depend on the n-point correlation functions for $n > 2$, and all show remarkable agreement with the corresponding random-matrix forms [28]. For example, this is the case for the distribution of spacings between adjacent unfolded zeros, which depends on all of the n-point correlation functions.

The conclusion to be drawn is that the statistical distribution of the Riemann zeros, in the limit as one looks infinitely high up the critical line, coincides with the statistical distribution of the eigenvalues of random unitary matrices, in the limit of large matrix size. (A great deal is also known about the way in which zero statistics asymptotically approach the large-height limit described by random matrix theory – see, for example, [3, 7, 23, 4] – but this will not directly concern us here.)

2 Random matrices and $\log\zeta(1/2+it)$

Having described the connections between the zeros of the Riemann zeta function and the eigenvalues of random matrices, I now turn to the question of the value distribution of the zeta function itself on the critical line, or rather, to begin with, the logarithm of the zeta function on this line.

$\log\zeta(1/2+it)$ is a complex function of the height t up the critical line. An obvious question is: how are the real and imaginary parts of it distributed as t varies? In the limit as $t\to\infty$, the answer to this question is provided by a beautiful theorem due to Selberg [30, 28]: for any rectangle $B\in\mathbb{C}$,

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\left|\left\{t: T\le t\le 2T, \frac{\log\zeta(1/2+it)}{\sqrt{\frac{1}{2}\log\log T}}\in B\right\}\right| \\ = \frac{1}{2\pi}\int\int_B e^{-(x^2+y^2)/2}dx\, dy;\end{aligned} \tag{12}$$

that is, in the limit as T, the height up the critical line, tends to infinity, the value distributions of the real and imaginary parts of $\log\zeta(1/2+iT)/\sqrt{(1/2)\log\log T}$ each tend, independently, to a Gaussian with unit variance and zero mean. Crucially, Odlyzko's computations for these distributions when $T\approx t_{10^{20}}$ show significant systematic deviations from this limiting form [28]. For example, increasing moments of both the real and imaginary parts diverge markedly from the Gaussian values. There is, of course, no contradiction; this merely suggests that the limiting Gaussian distribution is approached rather slowly as $T\to\infty$. It does, though, lead to the question of how to model the statistical properties of $\log\zeta(1/2+it)$ when t is large but finite.

Given its success in describing the statistical properties of the zeros of the zeta function, it is natural ask whether random matrix theory might be used as the basis of such a model. The question is, then: what property of a matrix plays the role of the zeta function? The answer is simple: since the zeros of the zeta function are distributed like the eigenvalues of a random unitary matrix, the zeta function might be expected to be similar, in respect of its value distribution, to the function whose zeros are the eigenvalues, that is, to the *characteristic polynomial* of such a matrix. This idea was introduced and investigated in detail in [24]. My aim here is to provide an overview of some of the main results.

The characteristic polynomial of a unitary matrix A may be defined by

$$Z(A,\theta)=\det(I-A\mathrm{e}^{-i\theta}). \tag{13}$$

The moment generating function for $\mathrm{Re}\log Z$, for example, is thus

$$M_U(s;N)=\int_{U(N)}\exp(s\mathrm{Re}\log Z)dA=\int_{U(N)}|Z(A,\theta)|^s dA, \tag{14}$$

where the average over A is, as before, computed with respect to Haar measure on $U(N)$. Obviously Z may be written in terms of the eigenangles of A:

$$Z(A,\theta)=\prod_{n=1}^{N}\left(1-\mathrm{e}^{i(\theta_n-\theta)}\right). \tag{15}$$

Haar measure on $U(N)$ may also be expressed in terms of these eigenangles [31], allowing one to write

$$\int_{U(N)}|Z(A,\theta)|^s dA=\frac{1}{(2\pi)^N N!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<m\le N}\left|\mathrm{e}^{i\theta_j}-\mathrm{e}^{i\theta_m}\right|^2\left|\prod_{n=1}^{N}(1-\mathrm{e}^{i(\theta_n-\theta)})\right|^s d\theta_1\cdots d\theta_N. \tag{16}$$

This N-dimensional integral may then be computed by relating it to an integral evaluated by Selberg [26], giving

$$M_U(s;N) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{(\Gamma(j+s/2))^2}. \tag{17}$$

All information about the value distribution of $\mathrm{Re}\log Z$ is contained within (17): moments may be computed in terms of the derivatives of $M_U(s;N)$ at $s=0$, and the value distribution itself is the fourier transform of $M_U(iy,N)$. In the same way, information about the value distribution of $\mathrm{Im}\log Z$, and the joint value distribution of the real and imaginary parts of $\log Z$ may be computed. This leads to a central limit theorem for $\log Z$ [24] (see also [14, 1]): for any rectangle $B \in \mathbb{C}$

$$\lim_{N\to\infty} \text{meas.}\left\{ A \in U(N) : \frac{\log Z}{\sqrt{\frac{1}{2}\log N}} \in B \right\} = \frac{1}{2\pi}\int\int_B e^{-(x^2+y^2)/2}dxdy. \tag{18}$$

This theorem corresponds precisely to Selberg's for the value distribution of $\log\zeta(1/2+it)$, suggesting that random matrix theory, in the limit as the matrix-size tends to infinity, can indeed model the value distribution of $\log\zeta(1/2+it)$ as $t\to\infty$. The question that remains is whether it can also model the asymptotic approach to the limit, that is, the value distribution when t is large but finite.

In order to relate the large-t asymptotics for the zeta function to the large-N asymptotics for the characteristic polynomials we need a connection between t and N. Note that the scaling in Selberg's theorem (12) and that in (18) coincide if we set

$$N = \log\frac{t}{2\pi}. \tag{19}$$

Such an identification is natural, because it corresponds to equating the mean density of the Riemann zeros at height t to the mean density of eigenphases for $N\times N$ unitary matrices, and these are the only parameters that appear in the connection between the respective statistics (cf. (2) and (4)). This therefore prompts the question as to whether the rate of approach to Selberg's theorem as $t\to\infty$ is related to that for (18) as $N\to\infty$ (which can be computed straightforwardly using (17)) if we make the identification (19).

As already noted above, Odlyzko's numerical computations of the value distribution of the zeta function near to the 10^{20}th zero show significant deviations from the Gaussian limit (12). The integer closest to $\log(t_{10^{20}}/2\pi)$ is $N=42$ ($t_{10^{20}}\approx 1.5202\times 10^{19}$). Does the value distribution of $\log Z$ for 42×42 random unitary matrices match his data?

Figure 2 shows the value distribution for $\mathrm{Re}\log\zeta(1/2+it)$, scaled as in (12), computed by Odlyzko [28], together with the value distribution for $\mathrm{Re}\log Z$, scaled as in (18), with respect to matrices taken from $U(42)$. Also shown is the Gaussian with zero mean and unit variance which represents the limit distribution in both cases (as $t\to\infty$ and $N\to\infty$ respectively). The negative logarithm of these curves is plotted in Figure 3, highlighting the behaviour in the tails. In order to quantify the data, the moments of the three distributions are listed in Table 1.

It is clear that random matrix theory provides an accurate description of the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$. It also models $\mathrm{Im}\log\zeta(1/2+it)$ equally well [24]. This then suggests that, statistically, the zeta function at a large height t up the critical line behaves like a polynomial of degree N, where t and N are related by (19); and, moreover, that the polynomial in question is the characteristic polynomial of a random unitary matrix.

Of course, specific properties of the zeta function would be expected to appear in the description of its value distribution. The point is that these contribute at lower order in the

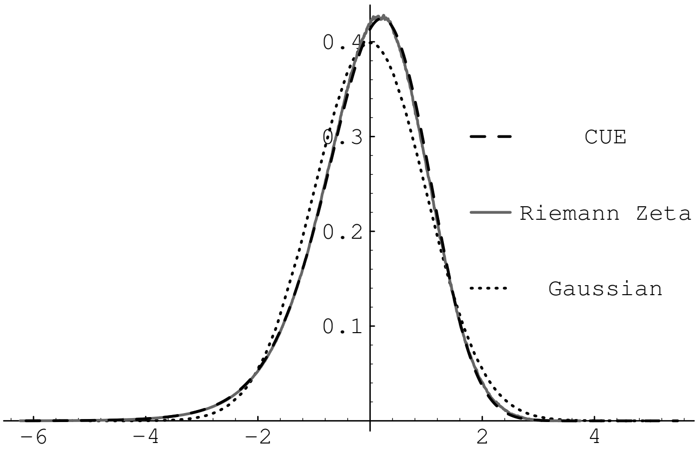

Figure 2: The value distribution for $\mathrm{Re}\log Z$ with respect to matrices taken from $U(42)$, Odlyzko's data for the value distribution of $\mathrm{Re}\log\zeta(1/2+it)$ near the 10^{20}th zero (taken from [28]), and the standard Gaussian, all scaled to have unit variance. (Taken from [24].)

Moment	ζ a)	ζ b)	$U(42)$	Normal
1	0.0	0.0	0.0	0
2	1.0	1.0	1.0	1
3	-0.53625	-0.55069	-0.56544	0
4	3.9233	3.9647	3.89354	3
5	-7.6238	-7.8839	-7.76965	0
6	38.434	39.393	38.0233	15
7	-144.78	-148.77	-145.043	0
8	758.57	765.54	758.036	105
9	-4002.5	-3934.7	-4086.92	0
10	24060.5	22722.9	25347.77	945

Table 1: Moments of $\mathrm{Re}\log\zeta(1/2+it)$, calculated over two ranges (labelled a and b) near the 10^{20}th zero ($t \simeq 1.520 \times 10^{19}$) (taken from [28]), compared with the moments of $\mathrm{Re}\log Z$ for $U(42)$ and the Gaussian (normal) moments, all scaled to have unit variance.

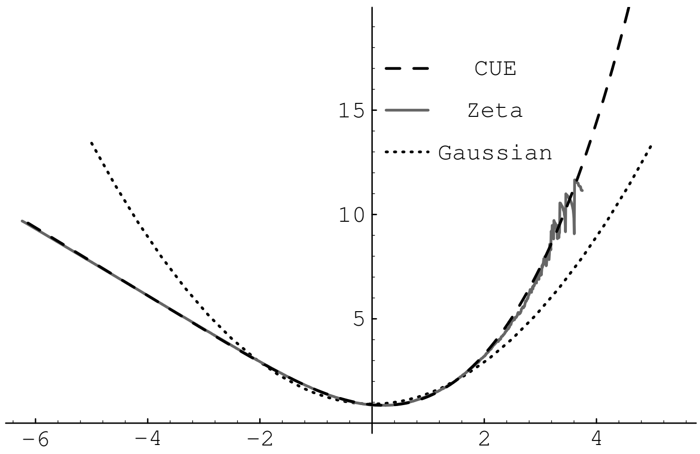

Figure 3: minus the logarithm of the value distributions plotted in Figure 2. (Taken from [24].)

asymptotics, with the leading order being given by random matrix theory. For example, it is shown in [24] that as $N \to \infty$

$$\int_{U(N)} (\mathrm{Im} \log Z)^2 dA = \frac{1}{2} \log N + \frac{1}{2}(\gamma + 1) + o(1), \tag{20}$$

where γ is Euler's constant, while Goldston [16] has proved, under the assumption of the Riemann Hypothesis and Montgomery's conjecture, that as $T \to \infty$

$$\begin{aligned} &\frac{1}{T} \int_0^T (\mathrm{Im} \log \zeta(1/2 + it))^2 dt \qquad (21) \\ &= \frac{1}{2} \log \log \frac{T}{2\pi} + \frac{1}{2}(\gamma + 1) + \sum_{m=2}^{\infty} \sum_p \frac{(1-m)}{m^2} \frac{1}{p^m} + o(1). \end{aligned}$$

These expression coincide under the identification (19), except for the sum over primes in (21). Obviously the primes have their origin in number theory, rather than random matrix theory.

3 Random matrices and $\zeta(1/2 + it)$

I now turn from the logarithm of the zeta function to the zeta function itself. Determining the value distribution of the zeta function is, it turns out, a significantly harder problem than determining the value distribution of its logarithm. Selberg's theorem completely characterizes the limiting distribution of $\log \zeta(1/2 + it)$, while for $\zeta(1/2 + it)$ almost nothing is known.

Regarding the moments of $|\zeta(1/2+it)|$, there is a long-standing and important conjecture that $f(\lambda)$ defined by

$$\lim_{T\to\infty} \frac{1}{\log^{\lambda^2}\left(\frac{T}{2\pi}\right)} \frac{1}{T} \int_0^T |\zeta(1/2+it)|^{2\lambda} dt = f(\lambda)a(\lambda), \tag{22}$$

where

$$a(\lambda) = \prod_p \left\{ (1-1/p)^{\lambda^2} \left(\sum_{m=0}^{\infty} \left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)} \right)^2 p^{-m} \right) \right\}, \tag{23}$$

exists, and a much-studied problem then to determine the values it takes, in particular for integer λ (see, for example, [30, 19]). Obviously $f(0) = 1$. In 1918, Hardy and Littlewood proved that $f(1) = 1$ [17], and in 1926 Ingham proved that $f(2) = 1/12$ [18]. No other values are known. Based on number-theoretical arguments, Conrey and Ghosh have conjectured that $f(3) = 42/9!$ [12], and Conrey and Gonek that $f(4) = 24024/16!$ [13].

Given the success of random matrix theory in describing the value distribution of $\log\zeta(1/2+it)$, it is natural to ask whether it has anything to contribute on this issue. Invoking the identification (19), the question for the characteristic polynomials that is analogous to (22) is whether

$$f_U(\lambda) = \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int_{U(N)} |Z(A,\theta)|^{2\lambda} dA \tag{24}$$

exists, and, if it does, what values it takes. The answer to this question was given in [24], where it was proved that f_U does indeed exist, that

$$f_U(\lambda) = \frac{G^2(1+\lambda)}{G(1+2\lambda)}, \tag{25}$$

where G denotes the Barnes G-function [2], and hence that $f_U(0) = 1$ (trivial) and

$$f_U(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \tag{26}$$

for integers $k \geq 1$. Thus, for example, $f_U(1) = 1$, $f_U(2) = 1/12$, $f_U(3) = 42/9!$ and $f_U(4) = 24024/16!$. The fact that these values coincide with those associated, or believe to be associated, with the zeta function strongly suggests that

$$f(\lambda) = f_U(\lambda) \tag{27}$$

for all $\mathrm{Re}\lambda > -1/2$. This conjecture is also supported by Odlyzko's numerical data for non-integer values of λ between zero and two [24]. (Conrey and Gonek's conjecture for $f(4)$ and ours for all integer λ were announced independently at the Erwin Schrödinger Institute in Vienna, in September 1998.)

4 L-functions

The zeta function is but one example of a more general class of functions known as L-functions. These all satisfy generalizations of the Riemann Hypothesis. For any individual L-function, it is believed that the zeros high up on the critical line are distributed like the eigenvalues of random unitary matrices, that is, exactly like in the case of the Riemann zeta function [27, 29]. This means that the moment conjecture described above generalizes immediately to all L-functions.

More interesting, however, is the fact that it has been conjectured by Katz and Sarnak [20, 21] that averages over various families of L-functions, with the height up the critical line of each one fixed, are described not only by averages over the unitary group $U(N)$, but by averages over other classical compact groups, for example the orthogonal group $O(N)$ or the unitary symplectic group $USp(2N)$, depending upon the family in question. This raises the important question of whether the moments of L-functions within these families can be determined by random matrix calculations generalizing those described above for the unitary group to the other classical compact groups [9]. The calculations for $O(N)$ and $USp(2N)$ were carried out in [25], where it was shown that the results agree with the few moments computed or conjectured using number-theoretical techniques.

5 Asymptotic expansions

The limit (22) may be thought of as representing the leading-order asymptotics of the moments of the zeta function, in that it implies that

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2\lambda}dt \sim f(\lambda)a(\lambda)\log^{\lambda^2}\left(\frac{T}{2\pi}\right) \tag{28}$$

as $T\to\infty$. Very little is known about lower order terms (in powers of $\log T$) in the asymptotic expansion of these moments. Does random matrix theory suggest what form these should take?

When λ is an integer, it does. Note first that it follows from (14) and (17) that

$$\int_{U(N)} |Z(A,\theta)|^{2k}dA = \prod_{j=0}^{k-1}\frac{j!}{(j+k)!}\prod_{i=1}^{k}(N+i+j) = Q_k(N) \tag{29}$$

where $Q_k(N)$ is a polynomial in N of degree k^2. This is consistent with the number theorists' guess that the $2k$th moment of the zeta function should be a polynomial of degree k^2 in $\log(T/2\pi)$ (modulo terms that vanish faster than any inverse power of $\log T/2\pi$ as $T\to\infty$).

Unfortunately it is not easy to see directly how to combine the coefficients in (29) with arithmetical information to guess the form of the coefficients of the lower-order terms in the moments of the zeta function. The expression in (29) can, however, be re-expressed in the form [10]

$$\int_{U(N)} |Z(A,\theta)|^{2k}dA = \frac{(-1)^k}{k!^2(2\pi i)^{2k}}\oint\cdots\oint e^{\frac{N}{2}\sum_{j=1}^{k} z_j - z_{j+k}}\frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}}dz_1\cdots dz_{2k}, \tag{30}$$

where the contours are small circles around the origin,

$$\Delta(z_1,\ldots,z_m) = \prod_{1\le i<j\le m}(z_j - z_i) \tag{31}$$

and

$$G(z_1,\ldots,z_{2k}) = \prod_{\substack{1\le \ell\le k\\ k+1\le q\le 2k}}(1-e^{z_q - z_\ell})^{-1}. \tag{32}$$

Note that G has simple poles when $z_i = z_j$, $i\neq j$. An evaluation of the contour integral in terms of residues confirms the identity by giving (29). This formula has a natural generalization to the zeta function [11]:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}dt = \frac{1}{T}\int_0^T W_k(\log\tfrac{t}{2\pi})(1+O(t^{-\frac{1}{2}+\epsilon}))dt, \tag{33}$$

where

$$W_k(x) = \frac{(-1)^k}{k!^2(2\pi i)^{2k}}\oint\cdots\oint e^{\frac{x}{2}\sum_{j=1}^k z_j - z_{j+k}}\frac{\tilde{G}(z_1,\dots,z_{2k})\Delta^2(z_1,\dots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}}\,dz_1\dots dz_{2k}, \tag{34}$$

the path of integration being the same as in (30), and

$$\tilde{G}(z_1,\dots,z_{2k}) = A_k(z_1,\dots,z_{2k})\prod_{i=1}^k\prod_{j=1}^k \zeta(1+z_i-z_{j+k}), \tag{35}$$

with

$$A_k(z) = \prod_p\prod_{i=1}^k\prod_{j=1}^k\left(1-\frac{1}{p^{1+z_i-z_{j+k}}}\right)\int_0^1\prod_{j=1}^k\left(1-\frac{e(\theta)}{p^{1/2+z_j}}\right)^{-1}\left(1-\frac{e(-\theta)}{p^{1/2-z_{j+k}}}\right)^{-1}d\theta \tag{36}$$

and $e(\theta)=\exp(2\pi i\theta)$. Note that $\tilde{G}$ has the same pole structure as G. An evaluation of this integral in terms of residues shows that W_k is a polynomial of degree k^2 and allows the coefficients to be computed. For example,

$$W_2(x) = 0.0506605918\,x^4 + 0.6988698848\,x^3 + 2.4259621988\,x^2 + 3.2279079649\,x + 1.3124243859 \tag{37}$$

and

$$W_3(x) = 0.0000057085\,x^9 + 0.0004050213\,x^8 + 0.0110724552\,x^7 + 0.1484007308\,x^6 + 1.0459251779\,x^5 + 3.9843850948\,x^4 + 8.6073191457\,x^3 + 10.2743308307\,x^2 + 6.5939130206\,x + 0.9165155076 \tag{38}$$

(we quote here numerical approximations for the coefficients, rather than the analytical expressions, which are rather cumbersome).

These polynomials describe the moments of the zeta function to a very high degree of accuracy [11]. For example, when $k=3$ and $T=2350000$, the left hand side of (33) evaluates to 1411700.43 and the right hand side to 1411675.64. Note that the coefficient of the leading order term is small. This explains the difficulties, described at length by Odlyzko [28], associated with numerical tests of (22).

Alternatively, one can also compare

$$\int_0^\infty |\zeta(1/2+it)|^{2k}\exp(-t/T)dt \tag{39}$$

with

$$\int_0^\infty W_k(\log(t/2\pi))\exp(-t/T)dt. \tag{40}$$

This is done in Table 2.

Similar asymptotic expansions have been derived for the moments of families of L-functions, using expressions analogous to (30) [11].

k	(39)	(40)	difference
1	79499.9312635	79496.7897047	3.14156
2	55088332.55512	55088336.43654	-3.8814
3	708967359.4	708965694.5	1664.9
4	143638308513.0	143628911646.6	9396866.4

Table 2: Smoothed moments, (39) and (40), when $T = 10000$

6 Conclusions

The conclusion one is led to draw from the results reviewed here is that random matrix theory, specifically results concerning the characteristic polynomials of random unitary matrices, leads to a conjectural solution, supported by all available evidence, to the long-standing problem of calculating the moments of the Riemann zeta function on its critical line. Moreover, this isn't an accident: the moments of families of other L-functions can be calculated using the same techniques. This opens up several major problems in number theory to investigation using methods developed to understand waves in complex systems (e.g. how the random matrix limit is approached asymptotically).

One obvious question it prompts is: what is the reason for the connection between random matrices and the zeta function? It has long been imagined there might be a spectral interpretation of the zeros. If the Riemann Hypothesis is true, such an interpretation could be the reason why; for example, if the zeros t_n are the eigenvalues of a self-adjoint operator, or the eigenphases of a unitary operator, then automatically they would all be real. Some speculations along these lines are reviewed in [4], others have been pursued by Connes and co-workers. If the zeros are indeed related to the eigenvalues of a self-adjoint or unitary operator, and if that operator behaves 'typically', this would then suggest that the zeros might be distributed like the eigenvalues of random matrices. The success of random matrix theory in describing properties of the zeta function might be interpreted as evidence in favour of a spectral interpretation.

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