Algebraic Geometry Lecture 1

Introduction by Dan Loughran

Let k be a field. We are interested in the ring of polynomials

$$A = k[x_1, \dots, x_n].$$

Given any polynomial $f \in A$ we can define the variety

$$V(f) = \{\underline{x} \in k^n : f(\underline{x}) = 0\}.$$

Example

$n = 1, f = a_n x^n + \dots + a_0$. Then $V(f) = \{\text{roots of } f\}$;
$n = 2, V(f)$ is a curve;
$n = 3, V(f)$ is a surface.

We define $\mathbb{P}^{n-1}$ to be the set of lines through the origin in $\mathbb{C}^n$ or $\mathbb{R}^n$, i.e. $\mathbb{C}^n / \sim$ where $\underline{a} \sim \underline{b}$ if and only if $\underline{a} = \lambda \underline{b}$ for some $\lambda \neq 0$. So one can think of

$$\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \qquad \mathbb{P}^2_{\mathbb{R}} = \mathbb{R} \cup \mathbb{P}^1.$$

Bézout's theorem If C_1 and C_2 are different curves of degrees d_1 and d_2 respectively then they intersect in exactly $d_1 d_2$ points in $\mathbb{P}^2$.

As number theorists we want rational solutions to the equation $f(x) = 0$ where

$$f(x) = a_d x^d + \dots + a_0.$$

Suppose we have a rational solution, say $f(p/q) = 0$, i.e.

$$a_d \frac{p^d}{q^d} + a_{d-1} \frac{p^{d-1}}{q^{d-1}} + \dots + a_1 \frac{p}{q} + a_0 = 0,$$

so that

$$a_d p^d + a_{d-1} p^{d-1} q + \dots + a_1 p q^{d-1} + a_0 q^d = 0.$$

Thence we have that $(p, q) \in \mathbb{Z}^2$ is a solution of

$$a_d X^d + a_{d-1} X^{d-1} Y + \dots + a_1 X Y^{d-1} + a_0 Y^d = 0.$$

Polynomials like these are called homogeneous. We study them in projective space because they satisfy the property

$$f(\lambda \underline{x}) = \lambda^d f(\underline{x}).$$

Let $f, g \in A$. Define a map

$$\varphi : V(f) \to V(g).$$

This is a *regular map* if

$$\varphi(a_1, \ldots, a_n) \mapsto (\varphi_1(a_1), \ldots, \varphi_n(a_n))$$

where the φ_i are polynomials. If the φ_i are rational functions then φ is called a *rational map.*

Example $f(x, y) = x^2 - y$. This curve is birationally equivalent to the line $y = 0$ by the map $\varphi(x, y) = (x, 0)$, i.e. projection down onto the x-axis.

If we call a curve the solution set of $f(x, y) = 0$ then each curve is a g-holed torus in $\mathbb{C}^2$. The value g is called the genus of the curve.

Genus g	Name	Examples	Properties	Number of rational points
0	Rational curves	Linear and quadratic polynomials	Birationally equivalent to $\mathbb{P}^1$	∞
1	Elliptic curves	$y^2 = x^3 + Ax + B$ with $4A^3 + 27B^2 \neq 0$	Actually abelian groups	$\leqslant \infty$, but finitely generated
$\geqslant 2$	General type	$x^n + y^n = z^n$ $n \geqslant 3$	They're difficult	$< \infty$