The Riemann zeta function and its moments

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Prime numbers

Definition

A *prime number* is a natural number which has exactly two distinct natural number divisors: 1 and itself.

Examples: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101...

Important problems

Let $\pi(x) := \#\{p \leq x \text{ prime}\}$.

- Is $\pi(x) \sim \mathrm{Li}\, x := \int_2^x \frac{1}{\log y}\, \mathrm{d}y$? [Yes, Prime Number Theorem]
- Is $\pi(x) - \mathrm{Li}\,(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$? [Open]
- Is $p_{n+1} - p_n = O\left(p_n^{\frac{1}{2}+\varepsilon}\right)$? [Open]

The Riemann zeta function

Definition

The *Riemann zeta function* is the function of a complex variable s, defined for $\Re(s) > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$.

It can be shown that $\zeta(s)$ can be extended to a meromorphic function defined over all complex values $s \neq 1$ in a unique way.

Important problems

- Is $\zeta(1 + it) \neq 0$ for all real t? [Yes]
- Do all the "non trivial" zeros of $\zeta(s)$ lie on the line $\Re(s) = \frac{1}{2}$? [Riemann hypothesis, open]
- Is $\zeta(\frac{1}{2} + it) = O(t^{\varepsilon})$? [Lindelöf hypothesis, open]

The connection between primes and $\zeta(s)$ I

Euler product:

$$\prod_{p \text{ prime}} (1-p^{-s})^{-1} = \prod_{p \text{ prime}} \left(\sum_{m=0}^{\infty} p^{-ms} \right) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s), \qquad \Re(s) > 1.$$

$$\begin{aligned}
\text{Infinitely many primes} \quad &\Longleftarrow \quad \lim_{y \to 1^+} \zeta(y) = +\infty \\
\text{Prime Number Theorem} \quad &\Longleftrightarrow \quad \zeta(1+it) \neq 0 \ \forall t \in \mathbb{R} \\
\pi(x) - \mathrm{Li}\,(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad &\Longleftrightarrow \quad \text{Riemann hypothesis} \\
p_{n+1} - p_n = O\left(p_n^{\frac{1}{2}+\varepsilon}\right) \quad &\Longleftarrow \quad \text{Lindelöf hypothesis}
\end{aligned}$$

The connection between primes and $\zeta(s)$ II

Chebyshev ψ function:

$$\psi(x) := \sum_{p^m \le x} \log p = \sum_{n \le x} \Lambda(n),$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, \\ 0 & \text{otherwise} \end{cases}$$

is the von Mangoldt function.
From the Euler product it's very easy to see that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

which implies

$$\int_1^x \psi(u)\,\mathrm{d}u = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \mathrm{d}s \qquad (x > 0, c > 1).$$

Some properties of $\zeta(s)$

Functional equation:

$$\zeta(1-s) := \chi(1-s)\zeta(s),$$

where

$$\chi(s) := 2(2\pi)^{-s}\Gamma(s)\cos\frac{\pi s}{2}.$$

Zero free region:

$$\zeta(\sigma+it) \neq 0 \quad \text{if} \quad \sigma > 1 - \frac{a}{\log(|t|+2)}, t \in \mathbb{R},$$

for some constant $a > 0$.

Moments of $\zeta(s)$

The $2k^{th}$ moment of the Riemann zeta function:

$$M_{2k}(T) := \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt$$

Why are they important?

- $M_{2k}(T) = O(T^{1+\varepsilon}) \iff$ Lindelöf hypothesis.
- Jensen's Formula and Littlewood's Lemma relate the number of zeros of a holomorphic function with moments.

Example:
At least two fifths of the zeros of $\zeta(s)$ are on the critical line.

Some results on moments

The asymptotic behaviour of $M_{2k}(T)$ is known just for $k = 1, 2$:

$$M_{2k}(T) := \int_1^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \sim \begin{cases} T \log T & \text{if } k = 1, \\ \frac{T}{2\pi^2}(\log T)^4 & \text{if } k = 2. \end{cases}$$

For $k > 2$ we don't have theorems, but just the conjecture

$$M_{2k}(T) \sim f_k a_k T \log^{k^2} T,$$

where a_k is a factor defined as an infinite product over primes and f_k is a rational number that can be predicted using Random Matrix Theory.

Shifted moments

The second shifted moment of $\zeta(s)$ is

$$I_{a,b}(T) := \int_0^T \zeta\left(\frac{1}{2}+a+it\right)\zeta\left(\frac{1}{2}-b-it\right)\,dt.$$

For bounded complex shifts a, b, we have

$$I_{a,b}(T) \sim \int_1^T \left(\zeta(1+a-b) + \left(\frac{t}{2\pi}\right)^{-a+b}\zeta(1-a+b)\right)\,dt.$$

Theorem

Let $|\Re(a)|, |\Re(b)| = O\left(\frac{1}{\log T}\right)$ *and* $|\Im(a)|, |\Im(b)| = O\left(T^{2-\varepsilon}\right)$.
Then

$$I_{a,b} \sim \int_0^T \zeta(1+a-b) + \chi(\frac{1}{2}+a+it)\chi(\frac{1}{2}-b-it)\zeta(1-a+b)\,dt.$$