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Some cases of Wilkie's conjecture (proofless version)

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Abstract

We estimate the number of algebraic points up to a given height on exponential-algebraic curves and a selection of exponential-algebraic surfaces, confirming a conjecture of Wilkie in their case.

1. *Introduction*

Finding bounds for the density of algebraic points in sets defined by analytic means is a long standing problem with deep historic links to transcendental number theory and, more recently, with other Diophantine applications. The presence of many algebraic points would suggest an underlying and perhaps unexpected rigidity to the set, and their paucity can be equally revealing. The wealth of information available about a set just by knowing about the density of its algebraic points thus makes counting these points a problem worthy of our attention.

The notion motivating the results in this paper is that if a reasonable set $Y \subset \mathbb{R}^n$ contains many rational points then these points should in fact lie in semi-algebraic subsets of Y. Concurrently, once these semi-algebraic subsets are removed, the remaining transcendental part of Y should contain relatively few rational points. Clearly these statements' veracity depends on what we mean by 'many' and 'few' rational points, and what 'reasonable' sets are.

The first of these issues is the simpler one. We will ultimately be counting real algebraic points in subsets of $\mathbb{R}^2$ and $\mathbb{R}^3$, that is elements of these subsets all of whose coordinates are algebraic. Some of the sets in question will only contain finitely many such points, such as the graph of the exponential function $y = e^x$. Here the only algebraic point is $(0, 1)$, so counting all the points is actually possible. Another possibility is that the set contains infinitely many algebraic points, such as with the graph of $y = 2^x$. Hence we instead aim to get density results on the number of algebraic points by only counting the number of points whose height is less than some parameter T. Heights abound in Diophantine geometry, here we use the multiplicative height H. This will be defined in section 3 but for now it is enough to know that if a/b is a rational number in lowest terms with $b > 0$ then $H(a/b) = \max\{|a|, b\}$,

Given a set $Y \subseteq \mathbb{R}^n$ and a number field $F \subset \mathbb{R}$ we set $Y(F) = Y \cap F^n$ and are interested in the density of points in $Y(F)$ with respect to height, videlicet the cardinality of the set

$$Y(F, T) = \{(y_1, \ldots, y_n) \in Y(F) \,:\, H(y_i) \leqslant T, 1 \leqslant i \leqslant n\}.$$

This cardinality we denote

$$N(Y, F, T) = \mathrm{card}(Y(F, T)).$$

As alluded to above, if a set contains semi-algebraic subsets then these subsets may contain many algebraic points, that is $\gg T^\delta$ algebraic points of height less than T for some positive constant δ. But while most of the algebraic points in a set Y may be found in the semi-algebraic

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subsets of Y, these are the least interesting algebraic points. We are interested in the truly transcendental part of the problem, so we define the algebraic part of a set Y to be union of all connected semi-algebraic subsets of Y of positive dimension. We denote this union Y^{alg}, and then ignore it and concentrate on counting the density of points in $Y^{\text{trans}} = Y \setminus Y^{\text{alg}}$.

Since we consider a set with $\gg T^{\delta}$ algebraic points to contain 'many' algebraic points, it seems reasonable to consider a set with $\ll T^{\varepsilon}$ algebraic points to contain 'few' algebraic points, where ε may be any arbitrarily small positive number. Thus we would hope that our reasonable sets Y satisfy

$$N(Y^{\text{trans}}, F, T) \ll_{F,\varepsilon,Y} T^{\varepsilon}.$$

Results of this ilk date back some time. For example, if we take Y to be the graph of some transcendental, analytic function $f : [0, 1] \to \mathbb{R}$, then Pila proved in [**Pil91**] that $N(Y, \mathbb{Q}, T) \ll_{\varepsilon,Y} T^{\varepsilon}$ for any $\varepsilon > 0$. Conversely, it was shown in [**Pil04**] that no better bound should be expected for general transcendental, analytic functions. In particular there are functions for which a bound of $\ll (\log T)^c$ is not possible for any fixed constant c.

None of this is to say that non-algebraic functions must usually be transcendental at algebraic arguments. Indeed, it was shown by Weierstrass that there exist entire, transcendental functions f such that $f(x)$ is rational for any rational number x. More recently it has been shown that given any subset A of the algebraic numbers, there is an entire, transcendental function f such that f and all its derivatives take algebraic values at any argument from A. Details can be found in [**HMM08**]. These constructions necessarily involve producing algebraic numbers with large heights.

In [**Pil04**] a bound of the form $\ll T^{\varepsilon}$ was established for the rational points on compact, subanalytic surfaces in $\mathbb{R}^n$. In the same paper it was conjectured that such a bound should hold for any compact, subanalytic set in $\mathbb{R}^n$. The class of such sets is one example of an o-minimal structure. This is an essentially model-theoretic notion (see, for example, [**Mar02**] for the model theoretic concepts required, and [**vdD98**] for a thorough discussion of o-minimality). Here we give a more geometrically intuitive description.

Definition 1.1. A *structure* on $\mathbb{R}$ is a sequence $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ such that for each n,

(S1) $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$ that is closed under finite unions and complements;
(S2) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_p$ then $A \times B \in \mathcal{S}_{n+p}$;
(S3) $\mathcal{S}_n$ contains every semialgebraic subset of $\mathbb{R}^n$;
(S4) if $A \in \mathcal{S}_{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map on the first n coordinates, then $\pi(A) \in \mathcal{S}_n$.

If a set $A \subseteq \mathbb{R}^n$ is in $\mathcal{S}_n$ then we say A is a *definable set in* $\mathcal{S}$. If, moreover, $\mathcal{S}$ is a structure and it satisfies the following condition:

(S5) the boundary of every set $A \in \mathcal{S}_1$ is finite,

then we call $\mathcal{S}$ an o-minimal structure.

The 'o' in o-minimal stands for 'order'. Condition (S3) means that $\mathcal{S}$ contains the usual ordering $<$ on $\mathbb{R}$, in the sense that the set $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is definable. Moreover, if one can define the ordering then necessarily one will be able to define intervals in $\mathbb{R}$ and hence, using (S1), finite unions of intervals and points. Condition (S5) says these are the *only* definable sets in $\mathbb{R}$, and so the structure is minimal in that sense.

The sets definable in o-minimal structures turn out to behave 'reasonably' in the sense we desired earlier. Indeed, Pila and Wilkie proved the following in [**PW06**].

THEOREM 1.2 (Theorem 1.8, [**PW06**]). *Let $Y \subset \mathbb{R}^n$ be a definable set in an o-minimal structure and let $\varepsilon > 0$. Then there is a constant $c(Y, \varepsilon)$ such that*

$$N(Y^{\mathrm{trans}}, \mathbb{Q}, T) \leqslant c(Y, \varepsilon) T^{\varepsilon}.$$

The paradigm of an o-minimal structure is the collection of semialgebraic sets. For them it is proving condition (S4) that is difficult, and this was accomplished by the Tarski–Seidenberg theorem. From a model theoretic perspective, the semialgebraic sets are what you get if you are only allowed to define subsets of $\mathbb{R}^n$ using the usual logical connectives and quantifiers as well as addition, multiplication, and the ordering on $\mathbb{R}$. If one also allows the use of the exponential function then one arrives at the structure denoted $\mathbb{R}_{\exp}$. In this structure the definable sets are projections of the zero sets of exponential polynomials, i.e. functions of the form $P(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ for some $P \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Wilkie proved in [**Wil96**] that this structure is o-minimal, the hard part in this case being the proof that the sets are closed under complements. In the same paper he showed the structure was model complete, which grants us the concise definition of the definable sets just given.

The fact that $\mathbb{R}_{\exp}$ is o-minimal means that theorem 1.2 applies to its definable sets. However, because the exponential function and this resulting structure are particularly well behaved, Wilkie conjectured in [**PW06**] that a better bound might hold for sets definable in this structure. In particular we have the following conjecture of Wilkie.

CONJECTURE 1.3 (Conjecture 1.11 of [**PW06**]). *Let Y be a definable set in $\mathbb{R}_{\exp}$. Then there are constants $c_1(Y)$ and $c_2(Y)$ such that, for any $T \geqslant e$,*

$$N(Y^{\mathrm{trans}}, \mathbb{Q}, T) \leqslant c_1(Y)(\log T)^{c_2(Y)}.$$

Theorem 1.2 was extended to allow any real number field F in place of $\mathbb{Q}$ in [**Pil09b**], and so Wilkie's conjecture, if it holds in $\mathbb{Q}$, should be expected to hold with $\mathbb{Q}$ replaced by any real number field, a conjecture made explicitly in [**Pil09a**] where it is also conjectured that the exponent in the bound should be independent of the number field in question. In this more general form the conjecture was affirmed in [**Pil09a**] for the exponential-algebraic surface X defined by

$$X = \{(x, y, z) \in (0, \infty)^3 \, : \, \log x \log y = \log z\}.$$

That this is definable in $\mathbb{R}_{\exp}$ is easily checked by noting that it is the projection on the first three coordinates of the set in $\mathbb{R}^6$ given by

$$\widetilde{X} = \{(x, y, z, u, v, w) \, : \, (x - e^u)^2 + (y - e^v)^2 + (z - e^w)^2 + (uv - w)^2 = 0\}.$$

In this paper we have two aims. First we will prove Wilkie's conjecture for any definable subset of $\mathbb{R}^2$ (note that the conjecture trivially holds for subsets of $\mathbb{R}$).

THEOREM 1.4. *Let $Y \subset \mathbb{R}^2$ be definable in $\mathbb{R}_{\exp}$, and let $f \in \mathbb{N}$. Then there are constants $c_1 = c_1(Y, f)$ and $c_2 = c_2(Y)$ such that, for any number field $F \subset \mathbb{R}$ of degree f and for any $T \geqslant e$,*

$$N(Y^{\mathrm{trans}}, F, T) \leqslant c_1(\log T)^{c_2}.$$

Following that we will establish a certain property called mildness, defined in section 4, of a family of functions, and consequently extend the results of [**Pil09a**] and confirm the conjecture for a family of surfaces in $\mathbb{R}^3$.

THEOREM 1.5. *Let $f \in \mathbb{N}$ and $a, b, c \in \mathbb{R}$. Define $Y \subset \mathbb{R}^3$ by*

$$Y = \{(x, y, z) \in (0, \infty)^3 \,:\, (\log x)^a (\log y)^b (\log z)^c = 1\}.$$

Then there are constants $c_3 = c_3(a, b, c, f)$ and $c_4 = c_4(a, b, c)$ such that for any number field $F \subset \mathbb{R}$ of degree f and any $T \geqslant e$,

$$N(Y^{\text{trans}}, F, T) \leqslant c_3 (\log T)^{c_4}.$$

In fact more is true. The exponent c_4 may be replaced by an absolute constant, independent of a, b, and c, granting uniformity across this family of surfaces.

These results may be considered as a midway point between the Pila–Wilkie result which gives a bound of T^{ε} on these sets, and Schanuel's conjecture in transcendental number theory which, for many of these sets, predicts only finitely many algebraic points. If a, b, c in theorem 1.5 are all integers, for example, then, pathological cases aside, Schanuel's conjecture predicts no nontrivial algebraic points in the surface Y.

The tools used for the proof of theorem 1.4 involve some minor model theory and then results developed in [**Pil07**] for counting rational points on curves, and extended in [**Pil09b**] for the algebraic case. The proof of theorem 1.5 uses the same tools as [**Pil09a**], the main difficulty lying in proving the surfaces can be parametrised by functions that are sufficiently well behaved to employ these tools.

Throughout the paper we denote the cardinality of a set Y by $\mathrm{card}(Y)$, we denote the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and the non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Definable sets will always be definable with parameters in the model theoretic sense.

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2. *Some model theoretic concepts*

One of the many useful properties of o-minimal structures is that of *cell decomposition*. That is, for any definable set $Y \subseteq \mathbb{R}^n$ there is a partition of $\mathbb{R}^n$ into a finite number of particularly nice subsets, called cells, such that for any one of these cells C, either $C \cap Y = \emptyset$ or $C \cap Y = C$. These cells are defined recursively on n.

In $\mathbb{R}$, cells are either points or intervals. Thus, by the defining property of o-minimal structures, given a definable set one can let the connected components be cells and the points and intervals between these connected components be cells, and one is done.

For higher dimensions we first need to define a *definable function* in a structure as a function $f : \mathbb{R}^m \to \mathbb{R}^n$ whose graph $\Gamma(f) \subset \mathbb{R}^{m+n}$ is a definable set. Then a cell in $\mathbb{R}^{n+1}$ is either the graph of a continuous definable function $f : C \to \mathbb{R}$ where C is a cell in $\mathbb{R}^n$, or it is a set of the form

$$(f, g)_C = \{(x, r) \in C \times \mathbb{R} \,:\, f(x) < r < g(x)\}$$

where again C is a cell in $\mathbb{R}^n$ and f and g are continuous definable functions, or, in this case, f may be $-\infty$ and g may be $+\infty$.

Above we only said f and g were continuous, but at the cost of having more cells one can let the functions be C^k for any $k \in \mathbb{N}$. In the structure we are interested in, $\mathbb{R}_{\exp}$, one can do even better than that. In this structure we have *analytic cell decomposition*, which, as the name suggests, means our functions may be analytic.

Thus in $\mathbb{R}^2$ every definable set Y is composed of a finite collection of graphs of analytic functions and open regions bounded by the graphs of analytic functions. We are interested in Y^{trans}, though, so we throw out all the semialgebraic open balls contained in an open set, necessarily leaving just graphs of analytic functions to worry about. We can also map everything into the unit box $[0,1]^2$ by means of the maps $\pm x^{\pm 1}$, which preserves both the algebraic parts of the set and the algebraic points and their heights. Finally we note that the intersection of Y with the boundary of this box will be a finite set of semialgebraic lines and points, the first of which we can throw away and the second of which we add to our eventual constant $c_1(Y)$.

So we may assume Y is a finite collection of graphs of analytic functions $\phi_i : (0,1) \to (0,1)$. Since the number of graphs is dependent only on Y there is no loss if we assume that Y is the graph of a single such function, and let the actual number of functions be included in the $c_1(Y)$ constant.

Another property of the structure $\mathbb{R}_{\exp}$ is that it is *model complete*. This term has a variety of equivalent definitions, but the one we need here is that any set $Y \subseteq \mathbb{R}^n$ that is definable in $\mathbb{R}_{\exp}$ is actually the image under a projection map $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ of the zero set of an exponential polynomial, that is a function of the form

$$P(x_1, \ldots, x_n, u_1, \ldots, u_m) = Q(x_1, \ldots, x_n, u_1, \ldots, u_m, e^{x_1}, \ldots, e^{x_n}, e^{u_1}, \ldots, e^{u_m}),$$

where $Q \in \mathbb{R}[t_1, \ldots, t_{2(n+m)}]$. Writing

$$V(P) = \{(x_1, \ldots, x_n, u_1, \ldots, u_m) \in \mathbb{R}^{n+m} \ : \ P(x_1, \ldots, x_n, u_1, \ldots, u_m) = 0\},$$

model completeness then means that

$$Y = \pi(V(P)).$$

Of course, the exponential polynomial P is determined by the set Y, so whenever characteristics of this function are used in the sequel (degree, etc.) we may consider them as constants depending on Y.

Our definable set is the graph of a definable, analytic function $\phi : (0,1) \to (0,1)$, and in this case we have an even more explicit description of the set from [**JW08**].

Theorem 2.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an analytic function, definable in $\mathbb{R}_{\exp}$. Then there are definable, analytic functions $\phi_2, \ldots, \phi_m : \mathbb{R} \to \mathbb{R}$, and exponential polynomials $f_1(x, y_1, \ldots, y_m), \ldots, f_m(x, y_1, \ldots, y_m)$ such that*

$$f_i(t, \phi(t), \phi_2(t), \ldots, \phi_m(t)) = 0$$

for $1 \leqslant i \leqslant m$, and

$$\det \left(\frac{\partial f_i}{\partial y_j} \right)_{1 \leqslant i,j \leqslant m} \neq 0.$$

Analytic cell decomposition and this consequence of model completeness are the only model theoretic aspects of the proof of theorem 1.4. The rest involves bounding the number of zeros of Pfaffian functions and their derivatives, and this is discussed in the next section.

3. *Proof of theorem 1.4*

The basis of the proof is a result of Pila that shows that one needs to do two things to count algebraic points on the graph of an analytic function: bound the number of zeros of the function's derivatives and the number of intersections of the graph with algebraic curves of a given degree. These two things are very possible if the function in question is Pfaffian.

Definition 3.1. A *Pfaffian chain* of order $r \geqslant 0$ and degree $\alpha \geqslant 1$ in an open domain $U \subseteq \mathbb{R}^n$ is a sequence of analytic functions $f_1, \dots, f_r$ in U satisfying the partial differential equations

$$\frac{\partial f_i}{\partial x_j} = p_{ij}(\vec{x}, f_1(\vec{x}), \dots, f_i(\vec{x})),$$

for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$, and where $p_{ij} \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_i]$ are polynomials of degree no greater than α. Given such a chain, a function

$$f(\vec{x}) = P(\vec{x}, f_1(\vec{x}), \dots, f_r(\vec{x})),$$

where $P \in \mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_r]$ is a polynomial of degree no greater than $\beta \geqslant 1$, is called a *Pfaffian function* of order r and degree (α, β).

Pfaffian functions, introduced by Khovanskiĭ, encompass a broad collection of functions, including the algebraic and elementary functions, and their various derivatives and compositions. Moreover, the order and degree (collectively referred to as the *format*) of the Pfaffian functions generated by differentiation and composition are expressible in terms of the format of the original functions. These functions have nice finiteness properties, in particular the following result.

Theorem 3.2 ([**GV04**], Corollary 3). *Let $f_1, \dots, f_m$ be Pfaffian functions in a "simple" domain $U \subseteq \mathbb{R}^n$, having a common Pfaffian chain, of order r and degrees (α, β_i) respectively. Let $\beta = \max_i \beta_i$ and let Y be the set*

$$Y = \{\vec{x} \in U \,:\, f_1(\vec{x}) = \ldots = f_m(\vec{x}) = 0\}.$$

Then Y has at most

$$2^{r(r-1)/2+1} \beta(\alpha + 2\beta - 1)^{n-1}((2n-1)(\alpha+\beta) - 2n + 2)^r$$

connected components.

The adjective simple in the statement of theorem 3.2 refers to generally nice subsets of $\mathbb{R}^n$ that don't cause the above bound to fail for pathological reasons, subsets like the positive orthant, the unit disc, the unit cube, and so on. All the domains we will use will be simple. When we use this bound, n, r, and α will be fixed and indeed only dependent on the set Y we are working with. Thus Y will have less than

$$c_1(Y)\beta^{c_2(Y)}$$

connected components.

The exponential function $\exp(x)$ is of course Pfaffian of order 1 and degree $(1, 1)$. Thus an exponential polynomial $Q(\vec{x}, e^{\vec{x}})$ in ℓ variables is a Pfaffian function of order ℓ and degree $(1, \deg Q)$.

As previously stated, we are counting points up to a certain value of their multiplicative height. This is defined as follows.

Definition 3.3. For an element x in a number field F, the *multiplicative height of* x is $H(x) = e^{h(x)}$, where $h(x)$ is the absolute logarithmic height

$$h(x) = \frac{1}{[F : \mathbb{Q}]} \sum_{v \in M_F} D_v \log^+ |x|_v.$$

Here $\log^+ x = \log\max\{1, x\}$, M_F is the set of places of F, and D_v is the local degree at v. These heights are independent of our choice of F.

The final ingredient for the proof of theorem 1.4 is the following result from [**Pil09b**].

THEOREM 3.4 ([**Pil09b**], Corollary 6.8). *Let $d \geqslant 1$, $D = (d+1)(d+2)/2$, $f \geqslant 1$, $T \geqslant e$, $T^{-4f} \leqslant L \leqslant 2T$, and $I \subset \mathbb{R}$ be an interval of length $\leqslant L$. Let $F \subset \mathbb{R}$ be a number field of degree $[F : \mathbb{Q}] = f$. Let $\phi : I \to \mathbb{R}$ have D continuous derivatives, with $|\phi'| \leqslant 1$ and $\phi^{(j)}$ either non-vanishing in the interior of I or identically vanishing, for $j = 1, \ldots, D$. Let Y be the graph of ϕ. Then $Y(F, T)$ is contained in the union of at most*

$$12f(4f+3)6^f DT^{4f(4f+1)/(3(d+3))} \log T$$

real algebraic curves (possibly reducible) of degree $\leqslant d$.

We are now ready to prove theorem 1.4 in the following form.

THEOREM 3.5. *Let $\phi : I \to (0,1)$ be an analytic function on some interval $I \subset (0,1)$ that is definable in $\mathbb{R}_{\exp}$ and let Y be its graph. Let $F \subset \mathbb{R}$ be a number field of degree $[F : \mathbb{Q}] = f$. Then there are constants $c_1 = c_1(Y, f)$ and $c_2 = c_2(Y)$ such that*

$$N(Y^{\mathrm{trans}}, F, T) \leqslant c_1(\log T)^{c_2}$$

for any $T \geqslant e$.

Proof. Proof omitted from this version. □

Note that the constant appearing in the exponent is independent of the degree of number field F suggesting some applications to transcendental number theory. Indeed, a trawl through the proof reveals that

$$c_8(Y) = 6m + 11,$$

so the exponent only depends on the number of exponential polynomials required in the result of theorem 2.1.

4. *Mild functions*

We now turn our attention to theorem 1.5. The strategy used here is related to that employed in proving theorem 1.4, but we also need to use the notion of mild functions. Through this section we will let $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{N}_0^k$, and use the Manhattan norm on elements of $\mathbb{N}_0^k$. For $\mu \in \mathbb{N}_0^k$ and $\phi : \mathbb{R}^k \to \mathbb{R}$, we write $\partial^\mu \phi$ for the partial derivative

$$\partial^\mu \phi = \frac{\partial^{|\mu|}\phi}{\partial x_1^{\mu_1} \cdots \partial x_k^{\mu_k}}.$$

We write $\vec{x}^\mu$ for the monomial $\prod x_i^{\mu_i}$ and let $\mu! = \prod \mu_i!$.

DEFINITION 4.1. Let $A, C \geqslant 0$. A function $\phi : (0,1)^k \to (0,1)$ is called (A, C)-*mild* if it is C^∞ and, for all $\mu \in \mathbb{N}_0^k$ and all $\vec{x} \in (0,1)^k$,

$$|\partial^\mu \phi(\vec{x})| \leqslant \mu!(A|\mu|^C)^{|\mu|}.$$

A function $\theta : (0,1)^k \to (0,1)^n$ is called (A, C)-mild if each of its coordinate functions is (A, C)-mild.

DEFINITION 4.2. A set $Y \subseteq (0,1)^n$ of dimension k is called (J, A, C)-*mild* if there is a finite collection Θ of (A, C)-mild maps $\theta : (0,1)^k \to (0,1)^n$ such that $\operatorname{card}\Theta = J$ and

$$\bigcup_{\theta\in\Theta} \theta((0,1)^k) = Y.$$

If a set Y is (J, A, C)-mild then we may just say it is mild, or has a mild parametrisation.

Mild functions coincide with Gevrey functions on $(0,1)^n$ (see, for example, [**YK06**]). But whereas Gevrey functions are usually classified by a single parameter, mild functions may be classified with two or even three parameters, affording more control in our results. In [**PW06**] a finite version of mildness was used to achieve the T^ε bound of theorem 1.2. It is conjectured in [**Pil09a**] that the intersection of any definable set in $\mathbb{R}_{\exp}$ with any set in an algebraic family of closed algebraic sets of given degree has a mild parametrisation with the mildness parameters depending only on the definable set and the degree of the algebraic sets. This would imply Wilkie's conjecture because of the following result.

THEOREM 4.3 ([**Pil09a**], corollary 3.3). *Suppose $Y \subset (0,1)^n$ of dimension k has a (J, A, C)-mild parametrisation. Let $F \subset \mathbb{R}$ be a number field of degree $[F : \mathbb{Q}] = f$. Then there is a constant $c(k, n)$ such that $Y(F, T)$ is contained in at most*

$$Jc(k,n)^f A^{(k+1)(1+o(1))}(f \log T)^{Cn(k+1)(1+o(1))/(n-k)}$$

intersections of Y with hypersurfaces of degree

$$d = \lfloor (f \log T)^{k/(n-k)} \rfloor.$$

Here, $1 + o(1)$ means as $T \to \infty$, with implicit constants depending on k and n.

To prove Wilkie's conjecture from this result, assuming the aforementioned conjecture, one simply intersects the original set $Y \subset \mathbb{R}^n$ with enough hypersurfaces to get all the algebraic points, and notes that these intersections form a definable set in $\mathbb{R}^{n-1}$. Repeating the procedure one eventually gets down to $\mathbb{R}^2$ and finds all the points.

At present is it not known that every set definable in $\mathbb{R}_{\exp}$ is mild. Certainly mildness is not a corollary of the underlying structure being o-minimal, as demonstrated by Margaret Thomas. However, the method just described for proving Wilkie's conjecture can still be used for particular sets, provided we can show that the sets in question are mild, and that intersecting them with hypersurfaces gives mild sets, and so on down to $\mathbb{R}^2$. In [**Pil09a**] this was done for the surface in $\mathbb{R}^3$ given by the equation $(\log x)(\log y) = \log z$. In this case, one inverts the part of the set that is without the unit cube to within, then intersecting the resulting surface with hypersurfaces gives sets that project down to Pfaff curves, whereupon the results of [**Pil07**] could be used. Since the surface has to be intersected with more and different hypersurfaces as the height bound increases theorem 1.4 can't replace this final step as it lacks sufficient uniformity. Thus we use the same techniques as those in [**Pil09a**].

Our set is defined by the equation

$$(\log x)^a(\log y)^b(\log z)^c = 1.$$

We won't lose any solutions if we just require the absolute value of the left-hand side to be 1, so we get the set defined by

$$|\log x|^a |\log y|^b |\log z|^c = 1.$$

Now if (x, y, z) satisfies this equation but any of x, y, or z is > 1 then we may replace that variable by its reciprocal and the modulus sign will get rid of the induced factor of -1. This introduces a small factor in our end estimate since we are essentially counting up to eight different points as the same one, but this can be included in the constant factor in our end result.

So theorem 1.5 boils down to the following.

THEOREM 4.4. *Let $a, b, c \in \mathbb{R}$ and define the set $Y \subset \mathbb{R}^3$ by*

$$Y = \{(x, y, z) \in (0, 1)^3 \,:\, |\log x|^a |\log y|^b |\log z|^c = 1\}.$$

Suppose Y has a mild parametrisation, then there are constants $c_1 = c_1(Y, f)$ and $c_2 = c_2(Y)$ such that

$$N(Y^{\text{trans}}, F, T) \leqslant c_1 (\log T)^{c_2}.$$

Proof. Proof omitted from this version. □

The above proof follows the same route as Pila's proof for the surface in [**Pil09a**]. The real work in extending this result is finding more mild functions to parameterise our surface. To complete the proof of theorem 1.5 we proceed to show Y has a mild parametrisation.

LEMMA 4.5. *let $p \in \mathbb{R}$ with $p \leqslant -1$. Then the function*

$$f(x) = \exp\left(-\left(\frac{1}{x} - 1\right)^p\right)$$

is mild.

Proof. Proof omitted from this version. □

If $p \geqslant 1$ then the function is simply a reflection in the line $x = 1/2$ of the case for $-p$. Clearly such a transformation has no effect on the level of mildness.

COROLLARY 4.6. *Let $p \in \mathbb{R}$ with $|p| \geqslant 1$. Then the function*

$$\exp\left(-\left(\frac{1}{x} - 1\right)^p\right)$$

is $(12|p|e^{\pi}, 2)$-mild.

The sum and product of mild functions are clearly mild, and an application of Faà di Bruno's formula is enough to show that the composition of mild functions is mild. For any $m \in \mathbb{N}_0^k$ the map $\mathbb{R}^k \to \mathbb{R}$ given by $\vec{x} \mapsto \vec{x}^m$ is mild.

COROLLARY 4.7. *Let $p \in \mathbb{R}$ satisfy $|p| \geqslant 1$ and $m \in \mathbb{N}_0^k$. Then the function*

$$\exp\left(-\left(\frac{1}{\vec{x}^m} - 1\right)^p\right)$$

is mild.

To complete the proof of theorem 1.5, then, we have the following.

LEMMA 4.8. *The set*

$$Y = \{(x, y, z) \in (0, 1)^3 \;:\; |\log x|^a |\log y|^b |\log z|^c = 1\}$$

has a mild parametrisation.

Proof. Proof omitted from this version. $\square$

The maps used to parameterise our surfaces are all (A, C)-mild where the value of A depends on the exponents a, b, c in the defining equation of the surface. But the value of C is wholly independent of the surface, granting us a uniformity across this family of surfaces.

COROLLARY 4.9. *Let $a, b, c \in \mathbb{R}$ and $f \in \mathbb{N}$, and define the set $Y \subset \mathbb{R}^3$ by*

$$Y = \{(x, y, z) \in (0, 1)^3 \;:\; |\log x|^a |\log y|^b |\log z|^c = 1\}.$$

There is a constant $c_1 = c_1(Y, f)$ and a fixed constant c_2 such that, for any number field F of degree f and any $T \geqslant e$,

$$N(Y^{\mathrm{trans}}, F, T) \leqslant c_1 (\log T)^{c_2}.$$

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