EXAMINATION SOLUTIONS

Linear Algebra and Geometry

January 2008

Any general comments on the examination should be written here...

1. (Standard problem)

   (a) (5 marks) The equation is equivalent to

   $$\frac{y^2}{16} - \frac{x^2}{9} = 1,$$

   which is an equation for a hyperbola with vertices $(0, \pm 4)$ and foci $(0, \pm 5)$.

   (b) (5 marks) Either replace (x, y) by $(x-1, y-6)$ (this shifts focus $(0, -5)$ to $(1, 1)$), or replace (x, y) by $(x-1, y+4)$ (this shifts focus $(0, 5)$ to $(1, 1)$). So one obtains the equation $9(y-6)^2 = 16(x-1)^2 + 144$, or $9(y+4)^2 = 16(x-1)^2 + 144$ (or an equivalent equation, such as $\frac{(y-6)^2}{16} - \frac{(x-1)^2}{9} = 1$ or $\frac{(y+4)^2}{16} - \frac{(x-1)^2}{9} = 1$).

2. (Standard problem)

   (a) (2 marks) $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$.

   (b) (4 marks) $7\mathrm{e}^{-i\pi/3} = 7/2 - i7\sqrt{3}/2$, so $a = 7/2$, $b = -7\sqrt{3}/2$.

   (c) (4 marks) $5\mathrm{e}^{i \cdot 3\pi/4} = -5\sqrt{2}/2 + i5\sqrt{2}/2$, so $c = -5\sqrt{2}/2$, $d = 5\sqrt{2}/2$.

3. (Bookwork)

   (a) (2 marks) $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if, for all $v, w \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, $T(v + w) = T(v) + T(w)$ and $T(\alpha v) = \alpha T(v)$.

   (b) (2 marks) $T : \mathbb{R}^n \to \mathbb{R}^m$ is injective if $T(v) \neq T(w)$ whenever $v \neq w$ $(v, w \in \mathbb{R}^n)$, or equivalently, if $T(v) = T(w)$ implies $v = w$ $(v, w \in \mathbb{R}^n)$.

   (c) (2 marks) $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective if for every $w \in \mathbb{R}^m$ there is some $v \in \mathbb{R}^n$ so that $T(v) = w$, or equivalently, if $T(\mathbb{R}^n) = \mathbb{R}^m$.

   (d) (2 marks) $\ker T = \{v \in \mathbb{R}^n : \ T(v) = 0 \ \}$.

   (e) (2 marks) The image of T is $\{T(v) : \ v \in \mathbb{R}^n \ \}$, or equivalently, the collection of all $w \in \mathbb{R}^m$ so that $w = T(v)$ for some $v \in \mathbb{R}^n$.

4. (Standard problem)

   (a) (5 marks) Subtract 5·row 2 from row 3 to get

   $$\begin{pmatrix} 0 & 1 & -2 \\ 1 & -4 & -4 \\ 0 & 3 & -3 \end{pmatrix}.$$

Subtract 3·row 1 from row 3 to get

$$\begin{pmatrix} 0 & 1 & -2 \\ 1 & -4 & -4 \\ 0 & 0 & 3 \end{pmatrix}.$$

Now swap rows 1 and 2 to get

$$\begin{pmatrix} 1 & -4 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

(b) (5 marks) $\det A = -1(-23+20) - 2(-17+20) = -3$.

5. (a) (8 marks) (Standard problem)

$$\begin{aligned} y - 2z &= -1 \\ x - 4y - 4z &= -1 \\ 5x - 17y - 23z &= 1 \end{aligned}$$

reduces to

$$\begin{aligned} x - 4y - 4z &= -1 \\ y - 2z &= -1 \\ 3z &= 9 \end{aligned}$$

So there is a unique solution: $z = 3, y = 5, x = 31$.

(b) (8 marks) (Standard problem)

$$\begin{aligned} x + 3y + 4z &= 8 \\ x + 5y + 8z &= 20 \\ 3x + 2y - 2z &= -18 \end{aligned}$$

reduces to

$$\begin{aligned} x + 3y + 4z &= 8 \\ 2y + 4z &= 12 \\ -7y - 14z &= -42 \end{aligned}$$

or equivalently

$$\begin{aligned} x + 3y + 4z &= 8 \\ y + 2z &= 6 \\ y + 2z &= 6 \end{aligned}$$

So there are infinitely many solutions: for any z, $y = 6 - 2z, x = 2z - 10$.

(c) (14 marks) (Partially unseen)

$$\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y - z &= -1 \end{aligned}$$

is equivalent to

$$\begin{aligned} x+y+kz &= 1 \\ (k-1)y+(1-k)z &= 0 \\ k(k+1)z &= k+1 \end{aligned}$$

If $k=0$ then there is no solution, since the last equation would be $0=1$. If $k=1$ then the second equation becomes $0=0$, the last equation becomes $2z=2$; so for any y, the solutions are $z=1, x=-y$. If $k=-1$ then the last equation becomes $0=0$, the second equation becomes $-2y+2z=0$, so for any z, the solutions are $y=z, x=1$. If k is not $0, \pm 1$ then we get a unique solution, $x=-1/k$, $y=z=1/k$.

6. (Standard problem)

   (a) (10 marks)
   $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(x,y) = (2y, x-y, 5x)$ is a linear transformation:

$$\begin{aligned} f((x,y)+(x',y')) &= f(x+x', y+y') \\ &= (2(y+y'), (x+x')-(y+y'), 5(x+x')) \\ &= (2y+2y', x-y+x'-y', 5x+5x') \\ &= (2y, x-y, 5x) + (2y', x'-y', 5x') \\ &= f(x,y)+f(x',y'). \end{aligned}$$

   Also,

$$\begin{aligned} f(\alpha(x,y)) &= f(\alpha x, \alpha y) \\ &= (2\alpha y, \alpha x - \alpha y, 5\alpha x) \\ &= \alpha(y, x-y, 5x) \\ &= \alpha f(x,y). \end{aligned}$$

   $g : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $g(x,y,z) = (3z, x-y^2)$ is not a linear transformation: for instance,

$$g((0,2,0)+(0,3,0)) = g(0,5,0) = (0,-25)$$

   whereas

$$g(0,2,0)+g(0,3,0) = (0,-4)+(0,-9) = (0,-13).$$

   $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x,y) = (y-1, x+3)$ is not a linear transformation: for instance,

$$h(2(1,1)) = h(2,2) = (1,5)$$

   whereas

$$2h(1,1) = 2(0,4) = (0,8).$$

   (Alternatively, one could use the vector $(0,0)$ rather than $(1,1)$ in this argument.)

   (b) (5 marks) f is given by the matrix

$$\begin{pmatrix} 0 & 2 \\ 1 & -1 \\ 5 & 0 \end{pmatrix}.$$

(c) (10 marks) Suppose T is injective. Thus if $T(v) = T(w)$ then $v = w$. We know $T(0) = 0$, so if $T(v) = 0$ then $v = 0$. So

$$\ker T = \{v \in \mathbb{R}^n : \ T(v) = 0 \ \} = \{0\}.$$

Now suppose $\ker T = \{0\}$. Say $T(v) = T(w)$. Then $0 = T(v) - T(w) = T(v - w)$; so $v - w \in \ker T$, which means $v - w = 0$, or in other words, $v = w$. Thus T is injective.

(d) (5 marks) Say $(x, y, z) \in \ker T$; then $x + 2z = 0, x + y = 0, 2x - y + 6z = 0$. This means $x = -y$, $x = -2z$ and $2x - y = 3x = -6z$. Hence $x = -y = -2z$, so $\ker T = \{(-2z, 2z, z) : \ z \in \mathbb{R} \ \}$.

7. (Standard problem, seen in lecture and lecture notes)

(a) (7 marks)

The line has direction $(6, 15, -16)$; the line consists of the points

$$\{r(6, 15, -16) + (1, -2, 5) : \ r \in \mathbb{R} \ \} = \{(6r + 1, 15r - 2, -16r + 5) : \ r \in \mathbb{R} \ \}.$$

(Alternatively, one could write this as $y = 15(x - 1)/6 - 2$, $z = -16(x - 1)/6 + 5$ where $x \in \mathbb{R}$.)

(b) (5 marks) $6x + 15y - 16z = 0$

(c) (10 marks) Let L denote the line in (a). Then

$$\begin{aligned} T(L) &= \{T(r(6, 15, -16) + (1, -2, 5)) : \ r \in \mathbb{R} \ \} \\ &= \{rT(6, 15, -16) + T(1, -2, 5) : \ r \in \mathbb{R} \ \} \\ &= \{rv' + w' : \ r \in \mathbb{R} \ \} \end{aligned}$$

where $v' = T(6, 15, -16)$ and $w' = (1, -2, 5)$. Since T is injective, $v' = T(6, 15, -16)$ is not 0, so $T(L)$ is the line with direction v' through the point w'.

(d) (8 marks) We need to define S to be a linear transformation that maps $(6, 15, -16)$ to 0; so for instance we define S by $S(x, y, z) = (15x - 6y, 0, 0)$.

End of solutions.