LINEAR ALGEBRA AND GEOMETRY REVIEW (§19-36)

Know the definitions: vector space, linear transformation, kernel, subspace, linear combination, span(U) (where $U \neq \emptyset$), linear dependence/independence, basis, dimension, rank and nullity of a linear transformation, isomorphism, rank of a matrix, linear operator, eigenvalue, eigenvector, characteristic polynomial, diagonalisable, inner product, orthogonal set, norm.

Basis examples of vector spaces: $\mathbb{R}^n$, $\{f : \mathbb{R} \to \mathbb{R}\}$, $M_{m\times n}(\mathbb{F})$ where $\mathbb{F}$ is a field (so $\mathbb{F}$ could be $\mathbb{R}$ or $\mathbb{C}$), {polynomials in X with coefficients in $\mathbb{F}$} where $\mathbb{F}$ is a field.

Main results: Throughout, suppose V, V' are vector spaces over some field $\mathbb{F}$, and $T : V \to V'$ is a linear transformation.

- A subset U of V is a subspace of V if and only if (1) $U \neq \emptyset$, (2) U is closed under vector addition, (3) U is closed under scalar multiplication.
- T is injective if and only if $\ker T = \{0\}$.
- $\ker T$ is a subspace of V.
- $T(V)$ is a subspace of V'.
- With $U \subseteq V$, $U \neq \emptyset$, span(U) is a subspace of V.
- If $U \subseteq V$ spans V, then $T(U)$ spans $T(V)$.
- A subset U of V is linearly independent if and only if: whenever $\alpha_1 u_1 + \cdots + \alpha_m u_m = 0$ for some $\alpha_i \in \mathbb{F}$ and some $u_i \in U$ (where $m \geq 1$ and the $u_1, \ldots, u_m$ are distinct), then we must have $\alpha_1 = \cdots = \alpha_m = 0$.
- A linearly independent subset U of V can be extended to a basis for V.
- A spanning set U for V contains a basis for V.
- Any two bases for V contain the same number of elements.
- Say $\dim V = n$. If U is a linearly independent subset of V with (exactly) n elements, then U is a basis for V. If U is a subset of V with (exactly) n elements and U spans V, then U is a basis for V.

- $\text{rank}T + \text{nullity}T = \dim V$.
- In the case $V = \mathbb{R}^n$, $V' = \mathbb{R}^m$, and $T : V \to V'$ is a linear transformation: (1) If $m > n$ then T is not surjective. (2) If $m < n$ then T is not injective. (3) If $m = n$ then T is injective exactly when T is surjective.
- Suppose $\dim V = n$. Then V is isomorphic to $M_{n\times 1}(\mathbb{F})$, and to $M_{1\times n}(\mathbb{F})$, and to $\mathbb{F}^n$.
- Suppose $\dim V = n$ and $\dim V' = m$. Then relative to bases for V and for V', we can represent T by an $m \times n$ matrix.
- Suppose $\dim V = \dim V' = n$. Suppose that, relative to some bases for V and V', A is the matrix representing T. The following are equivalent: (1) T is an isomorphism. (2) A is invertible. (3) $\det A \neq 0$.
- Suppose $V' = V$ and $\dim V = n$. Given a basis $\mathcal{B}$ for V, let A be the matrix representing T relative to $\mathcal{B}$. Fix $\lambda \in \mathbb{F}$. The following are equivalent: (1) $T(v) = \lambda v$ for some $v \neq 0$. (2) $(T - \lambda)(v) = 0$ for some $v \neq 0$. (3) $\ker(T - \lambda) \neq \{0\}$. (4) $T - \lambda$ is not injective. (5) $T - \lambda$ is not an isomorphism. (6) $\det(A - \lambda I) = 0$.
- Suppose $V' = V$. Then for $\lambda \in \mathbb{F}$, $U_\lambda = \{v \in V :\ T(v) = \lambda v\ \}$ is a subspace of V.
- Suppose $V' = V$, $\dim V = n$, and relative to some basis $\mathcal{B}$ for V, A is the matrix representing T. Then A is diagonalisable if and only if V has a basis $\mathcal{B}'$ consisting of eigenvectors of T. In the case such $\mathcal{B}'$ exists, let $\iota : V' \to V$ be the identity map, and let C be the matrix for ι relative to the bases $\mathcal{B}'$ for V', $\mathcal{B}$ for V. Then C^{-1} exists, and $C^{-1}AC$ is the matrix for $T = \iota^{-1} \circ T \circ \iota$ relative to the basis $\mathcal{B}'$. Thus $C^{-1}AC$ is diagonal.

From here forward, suppose $\mathbb{F} = \mathbb{R}$ and $\langle\, ,\, \rangle$ is an inner product on V.

- For $u, v, w \in V$, $\alpha \in \mathbb{R}$, $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$ and $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$.
- $\langle v, v\rangle = 0$ if and only if $v = 0$.
- If U is an orthogonal subset of V then U is linearly independent.
- If $\dim V < \infty$ then V has an orthogonal basis.
- For $v, w \in V$, $|\langle v, w\rangle| \leq ||v|| \cdot ||w||$ and $||v + w|| \leq ||v|| + ||w||$.